CONTINUED FRACTION PATTERNS FOR NUMBERS RELATED TO CARLITZ-DRINFELD EXPONENTIAL

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ABSTRACT

CONTINUED FRACTION PATTERNS FOR NUMBERS RELATED TO CARLITZ-DRINFELD EXPONENTIAL

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Dr. Dinesh Thakur has written several articles on exponential and continued fractions. These articles expand upon various number theory and function field concepts, such as standard facts and notation related to continued fractions, the Carlitz-Drinfeld exponential, and general and simple Hurwitz number expansions. In addition to proving the interesting patterns and properties related to these numbers, Dr. Thakur does several things. He introduces new terminology, shows that subtle pattern variations occur with cardinality q equal to two, and presents fancy numbers, such as $e/\overline{p}$, where $\overline{p}$ is the degree two prime.

This thesis explores two of these articles in depth, and showcases the beauty of the Carlitz-Drinfeld exponential in several ways. A new Mathematica 5.2 program is written to generate the quoted and formulated sequences, which are displayed with new "Curly Bracket" and "Colon" notations. Additionally, the correlation of the exponential to the Ruler Function is highlighted.

ACKNOWLEDGEMENTS

Working to complete a master's degree in mathematics has been an effort which has taken me six years to complete. Working on and completing this thesis has been the high point of earning this degree. There are many people I would like to thank and acknowledge.

First, I would like to thank my Santa Clara High School math teachers, Lawrence (Larry) Coffman and Andrew Crowley. Their interest in mathematics and adept teaching styles in the late 1960's enabled me to discover a love of and aptitude for mathematics.

I have appreciated all of the teachers I have had at San José State University. These include the teachers I met during my Bachelor of Art's program in the 1970's, and more recently those whose classes I have taken for the master's program in the 2000's.

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I received online technical assistance from several mathematicians and computer specialists in various newsgroups. I would like to especially thank the following three people:

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3. Lars Madsen (Department of Mathematical Sciences, Denmark): was invaluable in assisting me with revamping my thesis style from default LaTeX to APA style guidelines. Without his assistance in the comp.text.tex newsgroup, I am not sure that I would have met the submittal deadline, or ended up with a document with as smooth and consistent formatting. His posts were provided in early 2007.

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Last and most importantly, I would like to thank the author of these articles, Dr. Dinesh S. Thakur. First, I would like to thank him for writing a series of papers which has grabbed and kept my interest for over three years. The more I study and research the ideas he has put forth here, the more I want to learn and explore them. Secondly, I would like to thank him for his always friendly, prompt, and helpful manner. He was always an e-mail or phone call away from providing tips and assistance, which he did "With best wishes." Additionally, he kindly reviewed the final draft of this thesis prior to its submission to the thesis committee in early February 2007. He offered many valuable suggestions, and helped me brainstorm for the best title.

The number e has always been a fascination for me, from my early twenties. When I worked for General Electric in 1976, I wrote a Fortran program to calculate e to approximately 786 places, which was a detailed expansion for the time. I am still interested in the property or characteristic of transcendental numbers which causes some of them to have continued fraction expansions with recognizable patterns, and others not to.

I have attached a copy of the Fortran program in Appendix J, p. 115.

And, so, this thesis is dedicated —

To transcendental numbers, the number e,
and to all those who love them ... .

Diana L. Mecum
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CONTENTS

Chapter 1

INTRODUCTION

Dr. Thakur wrote three related articles on exponential and continued fractions. This thesis explores the first two of them. The third article, a sequel to the second, is listed here, but is outside the scope of this work.

The three articles are:

1. Continued Fraction for the Exponential for $\mathbb{F}_q$[t], (Thakur, 1992)

2. Exponential and Continued Fractions, (Thakur, 1996)

3. Patterns of Continued Fractions for the Analogues of e and Related Numbers in the Function Field Case, (Thakur, 1997)

Briefly, the first two articles describe and expand upon:

1. Basic function field definitions and constructs.

2. Product of monic irreducible polynomials of degree dividing i - $[i]$.

3. Value of the Carlitz factorial at q^i - D_i.

4. Standard facts and notation related to continued fractions.

5. Development of the Carlitz-Drinfeld exponential $e(z)$.

6. Results and examples of exponential functions worked out by Euler and Hurwitz.

7. Definitions of "negative reverse repetition" and "e-type."

8. Development and expansion of general and simple Hurwitz numbers for the exponential and Mobïus transformation.

9. Subtle patterns of sequence expansions when $q = 2$. One number which is developed in detail is $e/\overline{p}$, where $\overline{p} = t^2 + t + 1$, the degree two prime.

The reasons for writing a thesis on these articles include the following:

1. Learn about and discover the properties of the Carlitz-Drinfeld exponential $e(z)$ and other analogues in the function field case.

2. Make Dr. Thakur's proofs and explanations more readily understandable to the upper division college student. Work through the proofs and explanations line by line and either corroborate the step theoretically, or provide a specific computational example.

3. Write a Mathematica 5.2 program which generates computational examples of most of his ideas. Examine the sequence expansions for $q = 2$ and $q = 3$.

4. Develop notations for cardinalities $q = 2$ and $q = 3$, which allow the sequences to be easily read at length. "Curly Bracket" and "Colon" notations are introduced for this purpose.

5. Provide outside corroborating material, such as the section on the Ruler Function, Appendix K, p. 118, which speaks to the fundamental relevance of the $e(z)$ function.

As a side note, this thesis contains numerous original quotations and wording from Dr. Thakur's 1992 and 1996 articles. Each of these two articles has its own nomenclature and referencing style, and a third one is added here. To reconcile and synchronize these three styles, a "Quote Citation Cross-Reference Table" is provided on the next page.

For example, the continued fraction fact "$p_n/q_n = [a_0, ..., a_n]$," is referenced as "(B)" in the 1992 article, "1.1.2" in the 1996 article, and as "Chapter 2.2.1, p. 10" in this thesis. This causes confusion in reading, and this table assists with reference understanding.

With these articles, Dr. Thakur expands upon and proves the interesting patterns of the continued fraction expansions of numbers related to the Carlitz-Drinfeld exponential. His writings, and the correlation of this pattern to the Ruler Function, emphasize its inherent mathematical fundamental and cohesive properties.

Table 1.1: Quote Citation Cross-Reference Table

Reference in 1992 article	Reference in 1996 article	Thesis location	Descriptive title
	[BS]	(Baum & Sweet, 1976)	Article "Continued fractions of algebraic power series in characteristic 2"
	[DMP]	(Dekking et al., 1982)	Article "Folds"
	[GHR]	(Goss et al., 1992)	Book *The Arithmetic of Function Fields*
	[PS]	(Poorten & Shallit, 1992)	Article "Folded continued fractions"
	[S1]	(Shallit, 1979)	Article "Simple continued fractions for some irrational numbers, I"
	[T1]	(Thakur, 1992)	Article "Continued Fraction for the Exponential for $\mathbb{F}_q$[t]"
(A)	1.1.1	Chapter 2.2.1, p. 10	Continued fraction fact
(B)	1.1.2	Chapter 2.2.1, p. 10	Continued fraction fact
(C)	1.1.3	Chapter 2.2.1, p. 10	Continued fraction fact
(D)	1.1.4	Chapter 2.2.1, p. 10	Continued fraction fact
(E)	1.1.5	Chapter 2.2.1, p. 10	Continued fraction fact
	1.1.6	Chapter 2.2.1, p. 10	Continued fraction fact
Paragraph 2, p. 150	3.1.1	Chapter 2.1.2, p. 6	Product of monic irreducible polynomials of A of degree dividing i - $[i]$
Paragraph 2, p. 150	3.1.2	Chapter 2.1.3, p. 7	Value of the Carlitz factorial at q^i - D_i
	5.1	Chapter 3.3.1, p. 58	Special Phenomena
	5.2	Chapter 3.3.2, p. 60	Special Phenomena
	Section 4	Chapter 3.2, p. 26	The $\mathbb{F}_q[t]$ Case

Chapter 2

CONTINUED FRACTION FOR THE CARLITZ-DRINFELD EXPONENTIAL

Dr. Thakur sets out and accomplishes many tasks with the writing of his 1992 article. He gives several pertinent function field properties, and provides the definition the Carlitz-Drinfeld exponential $e(z)$ for $\mathbb{F}_q[t]$. He recalls standard facts and notation of continued fractions, and shows with Theorem 1, that the continued fraction expansion of the exponential $e(z)$ has an interesting pattern. He concludes the article with various remarks on generalized continued fractions.

2.1 Analogue of the Exponential

In this first section, Dr. Thakur defines function field terms, and introduces $[i]$ and D_i, the product of monic irreducible polynomials of degree dividing i, and the Carlitz factorial, respectively. He then expands upon the Carlitz-Drinfeld exponential $e(z)$.

2.1.1 Function Field Terms Defined

Dr. Thakur begins with definitions of many of the important terms that he will use in his pair of articles. Let $\mathbb{F}_q$ be a finite field of cardinality q and of characteristic p. T is a variable which is equivalent to saying that it is transcendental over $\mathbb{F}_q$.

Dr. Thakur's 1992 and 1996 articles use the variable q to represent the cardinality of a field. Note that $q = p^n$, where p is the characteristic of the field, and n is a positive integer. Since this thesis works with prime q, usually equal to 2 or 3, the terms "cardinality" and "characteristic" can be used interchangeably. For more clarification of "cardinality" and "characteristic," see Appendix B.3, p. 68.

He defines four analogues, which he goes into in more detail in his book, *Function Field Arithmetic*. (Thakur, 2004, p. 3). They are paraphrased here:

1. The polynomial ring $A = \mathbb{F}_q[t]$ is an analogue of the integer ring $\mathbb{Z}$;

2. The rational function field $K = \mathbb{F}_q(t)$ is an analogue of the rational number field $\mathbb{Q}$;

3. The Laurent series field $K_\infty = \mathbb{F}_q((1/t))$, (here $1/t$ is a parameter at infinity), is an analogue of the real number field $\mathbb{R}$, which is the completion of $\mathbb{Q}$ at the usual absolute value;

4. $\Omega = C_\infty$, which is defined as the completion of an algebraic closure of K_∞, is an analogue of the complex number field $\mathbb{C}$, which is an algebraic closure of $\mathbb{R}$. Ω is the 'smallest' extension which is both algebraically closed and complete.

Additionally, a function field is defined as:

> Unless we relax it explicitly, from now on, by a function field, we would mean a function field of one variable over a finite field, or more precisely, a finite extension of some $\mathbb{F}_q(t)$. (Thakur, 2004, p. 2)

2.1.2 Product of Monic Irreducible Polynomials of Degree Dividing i - [i]

Next, Dr. Thakur defines $[i]$. Let $i > 0$. Then:

> Let $[i] := T^{q^i} - T$. This is just the product of monic irreducible elements of A of degree dividing i. Note $[i+1] = [i]^q + [1] = [i] + [1]^{q^i}$. (Thakur, 1992, p. 150)

Side note: In $\mathbb{F}_q$, $q = 0$. For example, in $\mathbb{F}_2$, $2 = 0$. Also in $\mathbb{F}_2$, $1 = -1$, though this appears strange at first. Because of this, over $\mathbb{F}_q$, $(a+b)^q = a^q + b^q$. This is an important fact in the function field case.

So,

$$[i] = T^{q^i} - T$$
$$[i+1] = T^{q^{i+1}} - T$$
$$[i]^q + [1] = (T^{q^i} - T)^q + T^{q^i} - T = (T^{q^{i+1}} - T^q) + T^q - T$$
$$= T^{q^{i+1}} - T = [i+1]$$

Also,

$$[i] + [1]^{q^i} = T^{q^i} - T + (T^q - T)^{q^i} = T^{q^i} - T + (T^{q^{i+1}} - T^{q^i})$$
$$= T^{q^{i+1}} - T = [i+1]$$

2.1.3 Value of the Carlitz Factorial at q^i - D_i

Then, with the notation of David Goss,

1. Let $i > 0$. We set
   $[i] := T^{r^i} - T \in \mathbb{A}$.
2. We set $D_0 = 1$, and for $i > 0$,
   $D_i := [i][i-1]^r \cdots [1]^{r^{i-1}}$.
3. We set $L_0 = 1$, and for $i > 0$,
   $L_i := [i][i-1] \cdots [1]$. (Goss, 1996, p. 44)

The variable q is used in place of r here. The first few D_i terms are 1, [1], $[2][1]^q$, $[3][2]^q[1]^{q^2}$, $[4][3]^q[2]^{q^2}[1]^{q^3}$, ...

> The numbers $[i]$, D_i and L_i are fundamental for the arithmetic of $\mathbb{F}_r[t]$. (Goss, 1996, p. 44)

Dr. Thakur explains in his book, *Function Field Arithmetic*, that D_i is the value of the Carlitz factorial, which is an analog of the usual factorial at q^i. (Note, the 1992 article uses q^i. Dr. Thakur's book uses q^n. Also, the spelling of "analogue" in this thesis conforms to the specific reference, which may be "analogue" or "analog".)

> The defining polynomial $[n] = t^{q^n} - t$ of $\mathbb{F}_{q^n}$, which can also be described as the product of monic irreducible polynomials of degree dividing n, does sometimes play some role analogous to the usual n, or rather q^n.

> We have $q^n! = D_n = ([n] - [0])\,([n] - [1]) \cdots ([n] - [n-1])$ in this context looking like a factorial of $[n]$. There is also a q-twisted recursion analog of $n! = n(n-1)!$: $D_n = [n]D_{n-1}^q$. One has twisted recursions
>
> $$\prod(q^{n+1}) = [n+1]\prod(q^n)^q, \qquad [n+1] = [n]^q + [1]$$
>
> in place of the usual $(n+1)! = (n+1)n!$ and $(n+1) = n+1$ respectively.
>
> In this vein, note that $[n][n-1] \cdots [1] = L_n$. In fact, we can also think of 'factorial' of $n-k$ by product of differences as above, but in a backward manner: $([k+1] - [k])\,([k+2] - [k]) \cdots ([n] - [k]) = L_{n-k}^{q^k}$. (Thakur, 2004, p. 130)

The following exponential function of the Carlitz module is a special case of the Drinfeld module, and is an object of fundamental importance in function field arithmetic. This function, known as the Carlitz-Drinfeld exponential, is the topic of this thesis.

The exponential function is defined as follows:

> $$e(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i}.$$
>
> (Thakur, 1992, p. 150)

> $e(z)$ is an entire function (in the sense that the power series defining it converges for all $z \in \Omega$), but it is additive unlike the classical exponential, which is multiplicative. (Thakur, 1992, ps. 150, 151)

The motivation for defining the exponential function in this way is as follows.

> The exponential function is one of the most fundamental functions. Let us try to see what its analog, if any, should be in the function field case. We start with the simplest case of $A = \mathbb{F}_q[t]$. By the basic analogies recalled in the first chapter, we would like to have analogue $e(z)$ with range and domain C_∞. (Thakur, 2004, p. 31)

> So we seek the exponential function to be an entire (given by everywhere convergent power series) among those satisfying $e(x+y) = e(x) + e(y)$. (Thakur, 2004, p. 32)

> The resulting analog $e(z) : C_\infty \to C_\infty$ is an entire $\mathbb{F}_q$-linear function with linear term z and satisfies
>
> $$e(tz) = te(z) + e(z)^q.$$
>
> It is called the Carlitz exponential and the map $u \to tu + u^q$ defines Carlitz Module. (Thakur, 2004, ps. 32, 33)

The above quote states that $e(x + y) = e(x) + e(y)$. This is the definition of being "additive." In function fields, $f(a) = a^q$ is additive, and therefore, $e(z)$ is additive. More generally, for characteristic p, $f(a) = \sum a_i z^{p^i}$ is additive.

The first few terms of $e(z)$ look like the following:

$$e(z) = \frac{z}{1} + \frac{z^q}{D_1} + \frac{z^{q^2}}{D_2} + \frac{z^{q^3}}{D_3} + \frac{z^{q^4}}{D_4} + \cdots$$
$$= \frac{z}{1} + \frac{z^q}{[1]} + \frac{z^{q^2}}{[2][1]^q} + \frac{z^{q^3}}{[3][2]^q[1]^{q^2}} + \frac{z^{q^4}}{[4][3]^q[2]^{q^2}[1]^{q^3}} + \cdots.$$

This expands to:

$$e(z) = z + \frac{z^q}{T^q - T} + \frac{z^{q^2}}{(T^{q^2} - T)(T^{q^2} - T^q)} + \frac{z^{q^3}}{(T^{q^3} - T)(T^{q^3} - T^q)(T^{q^3} - T^{q^2})} + \cdots.$$

2.2 Continued Fractions

In this second section, Dr. Thakur recalls standard facts and notation of continued fractions. He then introduces and proves Theorem 1.

2.2.1 Standard Facts and Notation

Dr. Thakur first recalls standard facts and notation for continued fractions and the classical expansion of Euler's number e. He does this in both the 1992 and the 1996 article. He cites Hardy, (Hardy & Wright, 2000, p. 129).

> ... (we will use the abbreviation "CF" from now on for continued fraction) (Thakur, 1996, p. 249)

The standard facts and notation are as follows. The bullet labels and quotation from the 1996 article are used.

$[(a_i)] := [a_0,\ a_1,\ a_2,\ ...]$ denotes the continued fraction

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}.$$

The a_i is called the ith partial quotient. The quantity $a_n^{'}$ denotes $[a_n,\ a_{n+1},\ ...]$. By a tail of the CF $[(a_i)]$ we mean $a_n^{'}$ for some n.

1.1.1. Let

$$p_0 = a_0, \qquad q_0 = 1, \qquad p_1 = a_1 a_0 + 1, \qquad q_1 = a_1$$
$$p_n = a_n p_{n-1} + p_{n-2}, \qquad q_n = a_n q_{n-1} + q_{n-2}$$

1.1.2. $p_n/q_n = [a_0,\ ...,\ a_n]$,

1.1.3. $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$,

1.1.4. $[a_0,\ a_1,\ ...] = (a_n^{'} p_{n-1} + p_{n-2})/(a_n^{'} q_{n-1} + q_{n-2})$, and

1.1.5. $q_n/q_{n-1} = [a_n,\ ...,\ a_1]$.

1.1.6. For $x = [(x_i)]$ and $y = [(y_i)]$, $x_n^{'} = y_m^{'}$ for some m and n (i.e. tails agree) if and only if $y = (ax + b)/(cx + d)$ with $a,\ b,\ c,\ d \in \mathbb{Z}$ and $ad - bc = \pm\ 1$. We say that such x and y are equivalent. (Thakur, 1996, ps. 249, 250)

Observe that $[a + 1,\ b + 1] = [a,\ 1,\ b]$ in characteristic 2. (Thakur, 1996, p. 250)

A specific example follows.

..., Euler gave a nice continued fraction for e

$$[2,\ 1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1,\ 6,\ 1,\ ...,\ 1,\ 2n,\ 1,\ ...].$$

(Thakur, 1996, p. 248)

Then, for e, some explicit convergents are as follows:

$p_0 = 2,\ q_0 = 1$

$p_1 = 3,\ q_1 = 1$

$p_2 = 8,\ q_2 = 3$

$p_3 = 11,\ q_3 = 4$

$p_4 = 19,\ q_4 = 7$

$p_5 = 87,\ q_5 = 32$

$p_6 = 106,\ q_6 = 39$

$p_7 = 193,\ q_7 = 71$

$p_0/q_0 = 2 = [2]$

$p_1/q_1 = 3 = [2, 1]$

$p_2/q_2 = 8/3 = [2, 1, 2]$

$p_3/q_3 = 11/4 = [2, 1, 2, 1]$

$p_4/q_4 = 19/7 = [2, 1, 2, 1, 1]$

$p_5/q_5 = 87/32 = [2, 1, 2, 1, 1, 4]$

$p_6/q_6 = 106/39 = [2, 1, 2, 1, 1, 4, 1]$

$p_7/q_7 = 193/71 = [2, 1, 2, 1, 1, 4, 1, 1]$

$p_2q_1 - p_1q_2 = -1$

$p_3q_2 - p_2q_3 = 1$

$p_4q_3 - p_4q_2 = -1$

$p_5q_4 - p_4q_5 = 1$

$$[2, 1, 2, 1, 1, 4, 1, 1, 6, ...] = (a_3'p_2 + p_1)/(a_3'q_2 + q_1)$$
$$= ([1, 1, 4, 1, 1, 6, ...]\ (8) + (3))/([1, 1, 4, 1, 1, 6, ...]\ (3) + (1))$$
$$\approx ((1.81935)\ (8) + 3)/((1.81935)\ (3) + 1) \approx 2.71828$$

$$[2, 1, 2, 1, 1, 4, 1, 1, 6, ...] = (a_4'p_3 + p_2)/(a_4'q_3 + q_2)$$
$$= ([1, 4, 1, 1, 6, 1, ...]\ (11) + (8))/([1, 4, 1, 1, 6, 1, ...]\ (4) + (3))$$
$$\approx ((1.22048)\ (11) + 8)/((1.22048)\ (4) + 3) \approx 2.71828$$

$q_2/q_1 = 3 = [2, 1]$

$q_3/q_2 = 4/3 = [1, 2, 1]$

$q_4/q_3 = 7/4 = [1, 1, 2, 1]$

$q_5/q_4 = 32/7 = [4, 1, 1, 2, 1]$

2.2.2 Theorem 1

Dr. Thakur then states the main result of the 1992 article. He describes and proves the interesting pattern associated with the continued fraction expansion of the Carlitz-Drinfeld exponential $e(z)$.

THEOREM 1. Define a sequence x_n by setting $x_1 := [0, z^{-q}[1]]$ and if $x_n = [a_0, a_1, ..., a_{2^n-1}]$, then setting

$$x_{n+1} := [a_0, a_1, ..., a_{2^n-1}, -z^{-q^n(q-2)}D_{n+1}/D_n^2, -a_{2^n-1}, ..., -a_1],$$

then

$$x_n := \sum_{i=1}^{n} \frac{z^{q^i}}{D_i}.$$

In particular, $e(z) = z + \lim_{n\to\infty} x_n$ and the continued fraction for e is obtained by putting $z = 1$. In particular, for $q = 2$,

$$e = [1, \underbrace{\underbrace{\underbrace{[1]}, [2], \underbrace{[1]}}, [3], \underbrace{[1], [2], [1]}}, [4], \underbrace{[1], [2], [1], [3], [1], [2], [1]}, [5], ...]$$

$$e(z) = z + \frac{z^2}{[1]+\frac{z^2}{[2]+\frac{z^2}{[1]+\cdots}}}.$$

(Thakur, 1992, ps. 152, 153)

This sequence is derived in the following way. Note that the first term, for x_1 has 2 partial quotients: 0 and $z^{-q}[1]$. Each term x_n has 2^n partial quotients.

With $z = 1$ and $q = 2$, $x_1 = [0, 1^{-2}[1]] = [0, [1]]$ and $-z^{-q^n(q-2)} = -1$.

The calculation of x_2 follows. It will have 4 partial quotients.

$x_2 = [0, [1], -D_2/D_1^2, -[1]] = [0, [1], -\frac{[2][1]^2}{[1]^2}, -[1]]$

Note that:

$-\frac{[2][1]^2}{[1]^2} = -\frac{(T^{2^2}-T)(T^2-T)^2}{(T^2-T)^2} = -(T^4 - T) = -[2],$

so:

$x_2 = [0, [1], -[2], -[1]]$

Since the function field is $\mathbb{F}_2$, $1 = -1$, and,

$x_2 = [0, [1], -[2], -[1]] = [0, [1], [2], [1]]$

The calculation of x_3 follows. It will have 8 partial quotients.

$x_3 = [0, [1], [2], [1], -D_3/D_2^2, -[1], -[2], -[1]] = [0, [1], [2], [1], -\frac{[3][2]^2[1]^{2^2}}{([2][1]^2)^2},$
$-[1], -[2], -[1]]$

Note that:
$-\frac{[3][2]^2[1]^{2^2}}{([2][1]^2)^2} = -\frac{(T^{2^3}-T)(T^{2^2}-T)^2(T^2-T)^{2^2}}{((T^{2^2}-T)(T^2-T)^2)^2} = -(T^8 - T) = -[3],$

so:

$x_3 = [0, [1], [2], [1], [3], [1], [2], [1]]$

2.2.3 *Proof of Theorem 1*

Proof. Let $e_n := \sum_{i=1}^{n} z^{q^i}/D_i$ and $\overline{e}_n := e_n - 2z^{q^n}/D_n$. Also, for x_n as in the statement of the theorem, we let $\overline{x}_n := [0, -a_{2^n-1}, ..., -a_1]$. Let p_i, q_i, $\overline{p}_i$, $\overline{q}_i$ have the obvious meanings, corresponding to the continued fractions x_n, $\overline{x}_n$'s. (Note that the continued fraction for x_n is obtained by truncating that of x_{n+1}, but the analogous statement is not true for $\overline{x}_n$ and $\overline{x}_{n+1}$, so $\overline{p}_i$, $\overline{q}_i$ depend on the particular $\overline{x}_n$ we are considering). Let the induction hypothesis H_n be

$$q_{2^n-1} = -\overline{q}_{2^n-1} = z^{-q^n} D_n, \qquad p_{2^n-1} = e_n q_{2^n-1},$$
$$\overline{p}_{2^n-1} = \overline{e}_n \overline{q}_{2^n-1}.$$

Then, H_1 clearly holds. (Thakur, 1992, p. 153)

Note that $e_n = x_n$, as x_n is defined in Theorem 1.

See "Examples of Partial Quotients and Convergents for $e(z)$," Chapter 2.2.4, p. 18, below.

$e_1 = \frac{z^q}{D_1}, \qquad \overline{e}_1 = \frac{z^q}{D_1} - \frac{2z^q}{D_1} = -\frac{z^q}{D_1}$

$q_1 = -\overline{q}_1 = z^{-q}D_1$

$p_1 = e_1 q_1 = \frac{z^q}{D_1}\, z^{-q}D_1 = 1, \qquad \overline{p}_1 = \overline{e}_1\overline{q}_1 = (-\frac{z^q}{D_1})\,(-z^{-q}D_1) = 1$

$x_1 := [0,\, z^{-q}D_1] = \frac{z^q}{D_1}$

> Assume H_n. Applying (E) to the definitions of x_n, $\overline{x}_n$, we see that q_{2^n-2} is the same as $\overline{p}_{2^n-1}$, which is the same as $-p_{2^n-1} + 2$, by the induction hypothesis. (Thakur, 1992, p. 153)

q_{2^n-2} can be shown to equal $\overline{p}_{2^n-1}$ accordingly:

$q_{2^n-1}/q_{2^n-2} = [a_{2^n-1},\, a_{2^n-2},\, ...,\, a_1]$

$\overline{p}_{2^n-1}/\overline{q}_{2^n-1} = [0,\, -a_{2^n-1},\, -a_{2^n-2},\, ...,\, -a_1]$

Note, $[a_{2^n-1},\, a_{2^n-2},\, ...,\, a_1] = -\frac{1}{[0,-a_{2^n-1},-a_{2^n-2},...,-a_1]}$, so

$q_{2^n-1}/q_{2^n-2} = -\overline{q}_{2^n-1}/\overline{p}_{2^n-1} = q_{2^n-1}/\overline{p}_{2^n-1}$, and $q_{2^n-2} = \overline{p}_{2^n-1}$, as was to be shown.

To show $\overline{p}_{2^n-1} = -p_{2^n-1} + 2$, note the following:

$\overline{e}_n := e_n - 2\frac{z^{q^n}}{D_n}$

$2\frac{z^{q^n}}{D_n} = e_n - \overline{e}_n$

$q_{2^n-1} = z^{-q^n}D_n = \frac{1}{\frac{z^{q^n}}{D_n}} = \frac{2}{\frac{2z^{q^n}}{D_n}} = \frac{2}{e_n - \overline{e}_n}$

$q_{2^n-1}(e_n - \overline{e}_n) = 2$

$e_n q_{2^n-1} - \overline{e}_n q_{2^n-1} = 2$

$-\overline{e}_n q_{2^n-1} = -e_n q_{2^n-1} + 2$

$\overline{e}_n \overline{q}_{2^n-1} = -e_n q_{2^n-1} + 2$

$\overline{p}_{2^n-1} = -p_{2^n-1} + 2$

Thus, it has been shown that $\overline{p}_{2^n-1} = -p_{2^n-1} + 2$.

> Application of (C) then shows that $p_{2^n-2} = \frac{-(p_{2^n-1}-1)^2}{q_{2^n-1}}$. (Thakur, 1992, p. 153)

To show that $p_{2^n-2} = \frac{-(p_{2^n-1}-1)^2}{q_{2^n-1}}$, note the following. It has been shown that $q_{2^n-2} = -p_{2^n-1} + 2$. So:

$-q_{2^n-2} = p_{2^n-1} - 2$
$-p_{2^n-1}q_{2^n-2} = p_{2^n-1}(p_{2^n-1} - 2)$
$-p_{2^n-1}q_{2^n-2} = (p_{2^n-1})^2 - 2p_{2^n-1}$
$1 - p_{2^n-1}q_{2^n-2} = (p_{2^n-1})^2 - 2p_{2^n-1} + 1$
$1 - p_{2^n-1}q_{2^n-2} = (p_{2^n-1} - 1)^2$

Applying continued fraction fact 1.1.3, Chapter 2.2.1, p. 10, to the definition of x_n, it is seen that:

$p_{2^n-1}q_{2^n-2} - p_{2^n-2}q_{2^n-1} = (-1)^{2n-2} = 1$
$-p_{2^n-2}q_{2^n-1} = 1 - p_{2^n-1}q_{2^n-2}$
$p_{2^n-2} = -\frac{1-p_{2^n-1}q_{2^n-2}}{q_{2^n-1}}$

Using the fact that "$1 - p_{2^n-1}q_{2^n-2} = (p_{2^n-1} - 1)^2$," as shown above, it is seen that $p_{2^n-2} = -\frac{(p_{2^n-1}-1)^2}{q_{2^n-1}}$.

Thus, it has been shown that $p_{2^n-2} = -\frac{(p_{2^n-1}-1)^2}{q_{2^n-1}}$.

On the other hand, (D) together with H_n implies that

$$x_{n+1} = \frac{(-z^{-q^n(q-2)}D_{n+1}/D_n^2 + \overline{x}_n)p_{2^n-1} + p_{2^n-2}}{(-z^{-q^n(q-2)}D_{n+1}/D_n^2 + \overline{x}_n)q_{2^n-1} + q_{2^n-2}}.$$

(Thakur, 1992, p. 153)

Note that $(z^{-q^n(q-2)}D_{n+1}/D_n^2 + \overline{x}_n)$ is the tail of x_{n+1}, beginning with the middle-right, i.e., the $(2^n+1)^{th}$ partial quotient of x_{n+1}. This can be easily observed by studying the expression for "x_{n+1}" seen with the statement of Theorem 1 on page 12.

In consulting with Dr. Thakur in July 2006, it was confirmed that the term $z^{q^n(q-2)}D_{n+1}/D_n^2$ in the original article should be $z^{-q^n(q-2)}D_{n+1}/D_n^2$. This term is seen in the numerator and denominator of the expression for x_{n+1}, above.

Dr. Thakur next multiplies the numerator and denominator by $(-D_n/z^{q^n})$.

First, the numerator times $(-D_n/z^{q^n})$ is

$$((-z^{-q^n(q-2)}D_{n+1}/D_n^2+\overline{x}_n)p_{2^n-1}+p_{2^n-2})\;(-D_n/z^{q^n})$$

$$= (-z^{-q^n(q-2)}D_{n+1}/D_n^2)\;(p_{2^n-1})\;(-D_n/z^{q^n}) + (\overline{x}_n)\;(p_{2^n-1})\;(-D_n/z^{q^n}) + (p_{2^n-2})\;(-D_n/z^{q^n})$$

$$= (-z^{-q^n(q-2)}D_{n+1}/D_n^2)\;(p_{2^n-1})\;(-D_n/z^{q^n}) + (\tfrac{\overline{p}_{2^n-1}}{\overline{q}_{2^n-1}})\;(p_{2^n-1})\;(-D_n/z^{q^n}) + (p_{2^n-2})\;(-D_n/z^{q^n})$$

$$= (-z^{-q^n(q-2)}D_{n+1}/D_n^2)\;(p_{2^n-1})\;(-D_n/z^{q^n}) + (\tfrac{\overline{e}_n\overline{q}_{2^n-1}}{\overline{q}_{2^n-1}})\;(p_{2^n-1})\;(-D_n/z^{q^n}) + (-\tfrac{(p_{2^n-1}-1)^2}{q_{2^n-1}})\;(-D_n/z^{q^n})$$

$$= (-z^{-q^n(q-2)}D_{n+1}/D_n^2)\;(e_nq_{2^n-1})\;(-D_n/z^{q^n}) + (\overline{e}_n)\;(e_nq_{2^n-1})\;(-D_n/z^{q^n}) + (-\tfrac{(e_nq_{2^n-1}-1)^2}{q_{2^n-1}})\;(-D_n/z^{q^n})$$

$$= (z^{-q^n(q-2)}D_{n+1}/D_n^2)\;(e_nz^{-q^n}D_n)\;(D_n/z^{q^n}) + (e_n - 2z^{q^n}/D_n)\;(e_nz^{-q^n}D_n)\;(-D_n/z^{q^n}) + (\tfrac{(e_nz^{-q^n}D_n-1)^2}{z^{-q^n}D_n})\;(D_n/z^{q^n})$$

$$= z^{-q^n(q-2)-2q^n}D_{n+1}e_n + z^{-2q^n}D_ne_n(2z^{q^n}-D_ne_n) + (z^{-q^n}D_ne_n-1)^2$$

$$= z^{-q^{n+1}}D_{n+1}e_n + z^{-2q^n}D_ne_n(2z^{q^n}-D_ne_n) + (z^{-q^n}D_ne_n-1)^2$$

$$= 1 + z^{-q^{n+1}}D_{n+1}e_n = z^{-q^{n+1}}D_{n+1}e_{n+1}$$

The above line is true because of the following:

$$q_{2^{n+1}-1} = z^{-q^{n+1}}D_{n+1} = \frac{D_{n+1}}{z^{q^{n+1}}} = \frac{1}{\frac{z^{q^{n+1}}}{D_{n+1}}} = \frac{1}{e_{n+1}-e_n}$$

Since $q_{2^{n+1}-1} = \frac{1}{e_{n+1}-e_n}$,

$$z^{-q^{n+1}}D_{n+1}e_{n+1} - z^{-q^{n+1}}D_{n+1}e_n = 1$$

Likewise, the denominator times $(-D_n/z^{q^n})$ is

$$((-z^{-q^n(q-2)}D_{n+1}/D_n^2+\overline{x}_n)q_{2^n-1}+q_{2^n-2})\;(-D_n/z^{q^n})$$

$$= (-z^{-q^n(q-2)}D_{n+1}/D_n^2)\;(q_{2^n-1})\;(-D_n/z^{q^n}) + (\overline{x}_n)\;(q_{2^n-1})\;(-D_n/z^{q^n}) + (q_{2^n-2})\;(-D_n/z^{q^n})$$

$$= (-z^{-q^n(q-2)}D_{n+1}/D_n^2)\;(q_{2^n-1})\;(-D_n/z^{q^n}) + (\tfrac{\overline{p}_{2^n-1}}{\overline{q}_{2^n-1}})\;(q_{2^n-1})\;(-D_n/z^{q^n}) + (\overline{p}_{2^n-1})\;(-D_n/z^{q^n})$$

$$= (-z^{-q^n(q-2)}D_{n+1}/D_n^2)\;(q_{2^n-1})\;(-D_n/z^{q^n}) + (\tfrac{\overline{e}_n\overline{q}_{2^n-1}}{\overline{q}_{2^n-1}})\;(q_{2^n-1})\;(-D_n/z^{q^n}) + (\overline{e}_n\overline{q}_{2^n-1})\;(-D_n/z^{q^n})$$

$$= (-z^{-q^n(q-2)}D_{n+1}/D_n^2)\;(q_{2^n-1})\;(-D_n/z^{q^n}) + (\overline{e}_n)\;(q_{2^n-1})\;(-D_n/z^{q^n})$$

$$
\begin{aligned}
&\quad + (-\overline{e}_n q_{2^n-1})\ (-D_n/z^{q^n}) \\
&= (z^{-q^n(q-2)} D_{n+1}/D_n^2)\ (z^{-q^n} D_n)\ (D_n/z^{q^n}) \\
&\quad + (e_n - 2z^{q^n}/D_n)\ (z^{-q^n} D_n)\ (-D_n/z^{q^n}) \\
&\quad + (e_n - 2z^{q^n}/D_n)\ (z^{-q^n} D_n)\ (D_n/z^{q^n}) \\
&= z^{-q^n(q-2)-2q^n} D_{n+1} \\
&= z^{-q^{n+1}} D_{n+1}
\end{aligned}
$$

Accordingly,

$$x_{n+1} = \frac{z^{-q^{n+1}} D_{n+1} e_{n+1}}{z^{-q^{n+1}} D_{n+1}} = e_{n+1}.$$

From this evaluation of x_{n+1}, it is seen that $p_{2^{n+1}-1}$, $q_{2^{n+1}-1}$ are as stated in H_{n+1}, by an easy count, using (A), of their sign and degrees in T and z. Similarly, one sees using (D) and H_n, that

$$\overline{x}_{n+1} = \frac{(z^{-q^n(q-2)} D_{n+1}/D_n^2 + \overline{x}_n) p_{2^n-1} + p_{2^n-2}}{(z^{-q^n(q-2)} D_{n+1}/D_n^2 + \overline{x}_n) q_{2^n-1} + q_{2^n-2}}.$$

(Thakur, 1992, p. 153)

A similar manipulation is then performed for $\overline{x}_{n+1}$ as was done for x_{n+1}. The numerator and denominator of $\overline{x}_{n+1}$ is multiplied by $-D_n/z^{q^n}$. The difference in the results for $\overline{x}_{n+1}$ and x_{n+1} can be accounted for by the difference in the sign of the first term in the numerator and denominator.

$$
\begin{aligned}
&((z^{-q^n(q-2)} D_{n+1}/D_n^2 + \overline{x}_n) p_{2^n-1} + p_{2^n-2})\ (-D_n/z^{q^n}) \\
&= (z^{-q^n(q-2)} D_{n+1}/D_n^2)\ (p_{2^n-1})\ (-D_n/z^{q^n}) + (\overline{x}_n)\ (p_{2^n-1})\ (-D_n/z^{q^n}) \\
&\quad + (p_{2^n-2})\ (-D_n/z^{q^n}) \\
&= (z^{-q^n(q-2)} D_{n+1}/D_n^2)\ (p_{2^n-1})\ (-D_n/z^{q^n}) + (\tfrac{\overline{p}_{2^n-1}}{\overline{q}_{2^n-1}})\ (p_{2^n-1})\ (-D_n/z^{q^n}) \\
&\quad + (p_{2^n-2})\ (-D_n/z^{q^n}) \\
&= (z^{-q^n(q-2)} D_{n+1}/D_n^2)\ (p_{2^n-1})\ (-D_n/z^{q^n}) + (\tfrac{\overline{e}_n \overline{q}_{2^n-1}}{\overline{q}_{2^n-1}})\ (p_{2^n-1})\ (-D_n/z^{q^n}) \\
&\quad + (-\tfrac{(p_{2^n-1}-1)^2}{q_{2^n-1}})\ (-D_n/z^{q^n}) \\
&= (z^{-q^n(q-2)} D_{n+1}/D_n^2)\ (e_n q_{2^n-1})\ (-D_n/z^{q^n}) + (\overline{e}_n)\ (e_n q_{2^n-1})\ (-D_n/z^{q^n}) \\
&\quad + (\tfrac{(e_n q_{2^n-1}-1)^2}{q_{2^n-1}})\ (D_n/z^{q^n}) \\
&= (z^{-q^n(q-2)} D_{n+1}/D_n^2)\ (e_n z^{-q^n} D_n)\ (-D_n/z^{q^n})
\end{aligned}
$$

$$+\ (e_n - 2z^{q^n}/D_n)\ (e_n z^{-q^n} D_n)\ (-D_n/z^{q^n})$$

$$+\ (\tfrac{(e_n z^{-q^n} D_n - 1)^2}{z^{-q^n} D_n})\ (D_n/z^{q^n})$$

$$= -z^{-q^n(q-2)-2q^n} D_{n+1}e_n + z^{-2q^n} D_n e_n (2z^{q^n} - D_n e_n) + (z^{-q^n} D_n e_n - 1)^2$$

$$= -z^{-q^{n+1}} D_{n+1}e_n + z^{-2q^n} D_n e_n (2z^{q^n} - D_n e_n) + (z^{-q^n} D_n e_n - 1)^2$$

$$= 1 - z^{-q^{n+1}} D_{n+1} e_n$$

Similarly, the product of the denominator of $\overline{x}_{n+1}$ and $-D_n/z^{q^n}$ is $-z^{-q^{n+1}} D_{n+1}$.

Accordingly,

$$\overline{x}_{n+1} = \tfrac{1 - z^{-q^{n+1}} D_{n+1} e_n}{-z^{-q^{n+1}} D_{n+1}} = e_n - \tfrac{z^{q^{n+1}}}{D_{n+1}}$$

$$= e_n + \tfrac{z^{q^{n+1}}}{D_{n+1}} - \tfrac{2z^{q^{n+1}}}{D_{n+1}} = e_{n+1} - \tfrac{2z^{q^{n+1}}}{D_{n+1}} = \overline{e}_{n+1}.$$

> Similar manipulation then shows that $\overline{p}_{2^{n+1}-1}$, $\overline{q}_{2^{n+1}-1}$ are as stated in H_{n+1}, thus proving H_{n+1}. This completes the proof by induction. □ (Thakur, 1992, p. 153)

After the proof of Theorem 1, Dr. Thakur then finishes up the article with several remarks, including the following:

> *Remarks.* (1) In fact, the proof shows that the partial sums of any series of the form $1/r_1 - 1/(r_1^2 r_2) - 1/(r_1^4 r_2^2 r_3) + \cdots$ have continued fractions with similar "negative reverse repetition" with the terms r_1, r_2, r_3, ...
>
> (2) For $q = 2$, use of the auxiliary continued fraction $\overline{x}_n$ can be avoided and a simpler proof can be given by just using (E). (Thakur, 1992, p. 154)

In fact, related to item (2) above, for $q = 2$, $\overline{x}_n = x_n$ and the continued fraction for $\overline{x}_n$ will be obtained by truncating that of $\overline{x}_{n+1}$.

2.2.4 Examples of Partial Quotients and Convergents for e(z)

$x_n = [a_0, a_1, ..., a_{2^n-1}]$ $\qquad \overline{x}_n := [0, -a_{2^n-1}, ..., -a_1]$

Because of the way that the x_n and $\overline{x}_n$ terms are derived below, the only difference between them will be the sign of the middle-right, i.e., the $(2^{n-1}+1)^{th}$

partial quotient. The sign of the middle-right partial quotient for x_n will be negative, and the sign of the middle-right partial quotient for $\overline{x}_n$ will be positive.

$x_1 := [0,\, z^{-q}[1]] = 0 + \frac{1}{z^{-q}[1]} = \frac{z^q}{[1]} = \frac{z^q}{D_1} = \sum_{i=1}^{1} z^{q^i}/D_i$

$\overline{x}_1 := [0,\, -z^{-q}[1]] = -\frac{z^q}{D_1}$

$x_2 := [0,\, z^{-q}[1],\, -z^{-q(q-2)}D_2/D_1^2,\, -z^{-q}[1]] = \frac{(z^{-q}[1])(-z^{-q(q-2)}D_2/D_1^2)-1}{(z^{-q}[1])^2(-z^{-q(q-2)}D_2/D_1^2)}$

$= \frac{z^q}{D_1} + \frac{z^{q^2}}{D_2} = \sum_{i=1}^{2} z^{q^i}/D_i$

(Note: With Mathematica 5.2, FromContinuedFraction[$\{0,\, \alpha_1,\, \alpha_2,\, -\alpha_1\}$] gives $\frac{\alpha_1\alpha_2-1}{\alpha_1^2\alpha_2}$. All expressions in this subsection were simplified and corroborated with Mathematica 5.2.)

$\overline{x}_2 := [0,\, z^{-q}[1],\, z^{-q(q-2)}D_2/D_1^2,\, -z^{-q}[1]] = \frac{z^q}{D_1} - \frac{z^{q^2}}{D_2}$

Note that only the middle-right partial quotient changes sign between x_2 and $\overline{x}_2$.

$x_3 := [0,\, z^{-q}[1],\, -z^{-q(q-2)}D_2/D_1^2,\, -z^{-q}[1],\, -z^{-q^2(q-2)}D_3/D_2^2,\, z^{-q}[1],$
$z^{-q(q-2)}D_2/D_1^2,\, -z^{-q}[1]]$

$= \frac{(z^{-q}[1])^3 - \frac{(z^{-q}[1])^2}{-z^{-q(q-2)}D_2/D_1^2} - \frac{1}{(-z^{-q(q-2)}D_2/D_1^2)^2(-z^{-q^2(q-2)}D_3/D_2^2)}}{(z^{-q}[1])^4}$

$= \frac{z^q}{D_1} + \frac{z^{q^2}}{D_2} + \frac{z^{q^3}}{D_3} = \sum_{i=1}^{3} z^{q^i}/D_i$

(Note: With Mathematica 5.2, FromContinuedFraction[$\{0,\, \alpha_1,\, \alpha_2,\, -\alpha_1,\, \alpha_3,\, \alpha_1,\, -\alpha_2,\, -\alpha_1\}$] gives $\frac{\alpha_1^3 - \frac{\alpha_1^2}{\alpha_2} - \frac{1}{\alpha_2^2\alpha_3}}{\alpha_1^4}$.)

$\overline{x}_3 := [0,\, z^{-q}[1],\, -z^{-q(q-2)}D_2/D_1^2,\, -z^{-q}[1],\, z^{-q^2(q-2)}D_3/D_2^2,\, z^{-q}[1],$
$z^{-q(q-2)}D_2/D_1^2,\, -z^{-q}[1]]$

$= \frac{z^q}{D_1} + \frac{z^{q^2}}{D_2} - \frac{z^{q^3}}{D_3}$

$e_1 := \sum_{i=1}^{1} z^{q^i}/D_i = \frac{z^q}{D_1}$

$e_2 := \sum_{i=1}^{2} z^{q^i}/D_i = \frac{z^q}{D_1} + \frac{z^{q^2}}{D_2}$

$e_3 := \sum_{i=1}^{3} z^{q^i}/D_i = \frac{z^q}{D_1} + \frac{z^{q^2}}{D_2} + \frac{z^{q^3}}{D_3}$

$\overline{e}_1 := e_1 - 2z^q/D_1 = -\frac{z^q}{D_1}$

$\overline{e}_2 := e_2 - 2z^{q^2}/D_2 = \frac{z^q}{D_1} - \frac{z^{q^2}}{D_2}$

$\overline{e}_3 := e_3 - 2z^{q^3}/D_3 = \frac{z^q}{D_1} + \frac{z^{q^2}}{D_2} - \frac{z^{q^3}}{D_3}$

$q_0 = 1$

$q_1 = a_1 = z^{-q}D_1 = z^{-q}[1]$

$q_2 = a_2q_1 + q_0 = (-z^{-q(q-2)}D_2/D_1^2)\ (z^{-q}[1]) + 1 = 1 - \frac{z^{q-q^2}D_2}{D_1}$

$q_3 = a_3q_2 + q_1 = (-z^{-q}[1])\ (1 - \frac{z^{q-q^2}D_2}{D_1}) + z^{-q}[1] = z^{-q^2}D_2 = z^{-q^2}[2][1]^q$

$q_4 = a_4q_3 + q_2 = (-z^{-q^2(q-2)}D_3/D_2^2)\ (z^{-q^2}D_2) + 1 - \frac{z^{q-q^2}D_2}{D_1}$

$= 1 - \frac{z^{q-q^2}D_2}{D_1} - \frac{z^{q^2-q^3}D_3}{D_2}$

$q_5 = a_5q_4 + q_3 = (z^{-q}[1])\ (1 - \frac{z^{q-q^2}D_2}{D_1} - \frac{z^{q^2-q^3}D_3}{D_2}) + z^{-q^2}D_2$

$= z^{-q}D_1(1 - \frac{z^{q^2-q^3}D_3}{D_2})$

$q_6 = a_6q_5 + q_4$

$= (z^{-q(q-2)}D_2/D_1^2)\ (z^{-q}D_1(1 - \frac{z^{q^2-q^3}D_3}{D_2})) + 1 - \frac{z^{q-q^2}D_2}{D_1} - \frac{z^{q^2-q^3}D_3}{D_2}$

$= 1 - \frac{z^{q-q^3}D_3}{D_1} - \frac{z^{q^2-q^3}D_3}{D_2}$

$q_7 = a_7q_6 + q_5 = (-z^{-q}D_1)\ (1 - \frac{z^{q-q^3}D_3}{D_1} - \frac{z^{q^2-q^3}D_3}{D_2}) + z^{-q}D_1(1 - \frac{z^{q^2-q^3}D_3}{D_2})$

$= z^{-q^3}D_3 = z^{-q^3}[3][2]^q[1]^{q^2}$

$\overline{q}_i$ and $\overline{p}_i$ calculations done with $n = 3$.

Note that when $n = 3$, $q_7 = -\overline{q}_7 = z^{-q^3}D_3$.

$\overline{q}_0 = 1$

$\overline{q}_1 = \overline{a}_1 = z^{-q}[1]$

$\overline{q}_2 = \overline{a}_2\overline{q}_1 + \overline{q}_0 = (-z^{-q(q-2)}D_2/D_1^2)\ (z^{-q}[1]) + 1 = 1 - \frac{z^{q-q^2}D_2}{D_1}$

$\overline{q}_3 = \overline{a}_3\overline{q}_2 + \overline{q}_1 = (-z^{-q}[1])\ (1 - \frac{z^{q-q^2}D_2}{D_1}) + z^{-q}[1] = z^{-q^2}D_2$

$\overline{q}_4 = \overline{a}_4\overline{q}_3 + \overline{q}_2 = (z^{-q^2(q-2)}D_3/D_2^2)\ (z^{-q^2}D_2) + 1 - \frac{z^{q-q^2}D_2}{D_1}$

$= 1 - \frac{z^{q-q^2}D_2}{D_1} + \frac{z^{q^2-q^3}D_3}{D_2}$

$\overline{q}_5 = \overline{a}_5\overline{q}_4 + \overline{q}_3 = (z^{-q}[1])\ (1 - \frac{z^{q-q^2}D_2}{D_1} + \frac{z^{q^2-q^3}D_3}{D_2}) + z^{-q^2}D_2$

$= z^{-q}D_1(1 + \frac{z^{q^2-q^3}D_3}{D_2})$

$\overline{q}_6 = \overline{a}_6\overline{q}_5 + \overline{q}_4$

$$= (z^{-q(q-2)} D_2/D_1^2)\ (z^{-q} D_1(1 + \tfrac{z^{q^2-q^3} D_3}{D_2})) + 1 - \tfrac{z^{q-q^2} D_2}{D_1} + \tfrac{z^{q^2-q^3} D_3}{D_2}$$

$$= 1 + \tfrac{z^{q-q^3} D_3}{D_1} + \tfrac{z^{q^2-q^3} D_3}{D_2}$$

$$\overline{q}_7 = \overline{a}_7 \overline{q}_6 + \overline{q}_5$$

$$= (-z^{-q}[1])\ (1 + \tfrac{z^{q-q^3} D_3}{D_1} + \tfrac{z^{q^2-q^3} D_3}{D_2}) + z^{-q} D_1 (1 + \tfrac{z^{q^2-q^3} D_3}{D_2})$$

$$= -z^{-q^3} D_3 = -z^{-q^3} [3][2]^q [1]^{q^2}$$

$$p_0 = a_0 = 0$$

$$p_1 = e_1 q_1 = (\tfrac{z^q}{D_1})\ (z^{-q}[1]) = 1$$

$$p_2 = a_2 p_1 + p_0 = (-z^{-q(q-2)} D_2/D_1^2)\ (1) + 0 = -z^{-q(q-2)} D_2/D_1^2$$

$$p_3 = e_2 q_3 = (\tfrac{z^q}{D_1} + \tfrac{z^{q^2}}{D_2})\ (z^{-q^2} D_2) = 1 + \tfrac{z^{q-q^2} D_2}{D_1}$$

$$p_4 = a_4 p_3 + p_2 = (-z^{-q^2(q-2)} D_3/D_2^2)\ (1 + \tfrac{z^{q-q^2} D_2}{D_1}) + -z^{-q(q-2)} D_2/D_1^2$$

$$= -\ \tfrac{z^{-q(q-2)} D_2}{D_1^2} - \tfrac{z^{-q^2(q-2)} (1 + \frac{z^{q-q^2} D_2}{D_1}) D_3}{D_2^2}$$

$$p_5 = a_5 p_4 + p_3$$

$$= (z^{-q}[1])\ (-\ \tfrac{z^{-q(q-2)} D_2}{D_1^2} - \tfrac{z^{-q^2(q-2)} (1 + \frac{z^{q-q^2} D_2}{D_1}) D_3}{D_2^2}) + 1 + \tfrac{z^{q-q^2} D_2}{D_1}$$

$$= 1 - \tfrac{z^{-q(q-1)^2} D_1 D_3}{D_2^2} - \tfrac{z^{q^2-q^3} D_3}{D_2}$$

$$p_6 = a_6 p_5 + p_4$$

$$= (z^{-q(q-2)} D_2/D_1^2)\ (1 - \tfrac{z^{-q(q-1)^2} D_1 D_3}{D_2^2} - \tfrac{z^{q^2-q^3} D_3}{D_2}) - \tfrac{z^{-q(q-2)} D_2}{D_1^2}$$

$$-\ \tfrac{z^{-q^2(q-2)} (1 + \frac{z^{q-q^2} D_2}{D_1}) D_3}{D_2^2}$$

$$= -\ \tfrac{z^{-q^3} (z^{q^2} D_1 + z^q D_2)^2 D_3}{D_1^2 D_2^2}$$

$$p_7 = e_3 q_7 = (\tfrac{z^q}{D_1} + \tfrac{z^{q^2}}{D_2} + \tfrac{z^{q^3}}{D_3})\ (z^{-q^3} D_3) = 1 + \tfrac{z^{q-q^3} D_3}{D_1} + \tfrac{z^{q^2-q^3} D_3}{D_2}$$

$$\overline{p}_0 = \overline{a}_0 = 0$$

$$\overline{p}_1 = \overline{a}_1 \overline{a}_0 + 1 = (z^{-q}[1])\ (0) + 1 = 1$$

$$\overline{p}_2 = \overline{a}_2 \overline{p}_1 + \overline{p}_0 = (-z^{-q(q-2)} D_2/D_1^2)\ (1) + 0 = -z^{-q(q-2)} D_2/D_1^2$$

$$\overline{p}_3 = \overline{a}_3 \overline{p}_2 + \overline{p}_1 = (-z^{-q}[1])\ (-z^{-q(q-2)} D_2/D_1^2) + 1 = 1 + \tfrac{z^{q-q^2} D_2}{D_1}$$

$$\overline{p}_4 = \overline{a}_4 \overline{p}_3 + \overline{p}_2 = (z^{-q^2(q-2)} D_3/D_2^2)\ (1 + \tfrac{z^{q-q^2} D_2}{D_1}) + -z^{-q(q-2)} D_2/D_1^2$$

$$= -\ \tfrac{z^{-q(q-2)} D_2}{D_1^2} + \tfrac{z^{-q^2(q-2)} (1 + \frac{z^{q-q^2} D_2}{D_1}) D_3}{D_2^2}$$

$$
\begin{aligned}
\overline{p}_5 &= \overline{a}_5\overline{p}_4 + \overline{p}_3 = (z^{-q}[1])\,\Big(-\tfrac{z^{-q(q-2)}D_2}{D_1^2} + \tfrac{z^{-q^2(q-2)}(1+\frac{z^{q-q^2}D_2}{D_1})D_3}{D_2^2}\Big) + 1 + \tfrac{z^{q-q^2}D_2}{D_1} \\
&= 1 + \tfrac{z^{-q(q-1)^2}D_1D_3}{D_2^2} + \tfrac{z^{q^2-q^3}D_3}{D_2} \\
\overline{p}_6 &= \overline{a}_6\overline{p}_5 + \overline{p}_4 \\
&= (z^{-q(q-2)}D_2/D_1^2)\,\Big(1 + \tfrac{z^{-q(q-1)^2}D_1D_3}{D_2^2} + \tfrac{z^{q^2-q^3}D_3}{D_2}\Big) - \tfrac{z^{-q(q-2)}D_2}{D_1^2} \\
&\quad + \tfrac{z^{-q^2(q-2)}(1+\frac{z^{q-q^2}D_2}{D_1})D_3}{D_2^2} \\
&= \tfrac{z^{-q^3}(z^{q^2}D_1+z^qD_2)^2D_3}{D_1^2D_2^2} \\
\overline{p}_7 &= \overline{e}_3\overline{q}_7 = \Big(\tfrac{z^q}{D_1} + \tfrac{z^{q^2}}{D_2} - \tfrac{z^{q^3}}{D_3}\Big)\,(-z^{-q^3}D_3) = 1 - \tfrac{z^{q-q^3}D_3}{D_1} - \tfrac{z^{q^2-q^3}D_3}{D_2}
\end{aligned}
$$

Chapter 3

EXPONENTIAL AND CONTINUED FRACTIONS

Dr. Thakur addresses many topics with the writing of his 1996 article. He talks about the number e and Hurwitz numbers in the real number setting. He gives a background on continued fractions and defines the Carlitz-Drinfeld exponential $e(z)$ for $\mathbb{F}_q[t]$. He introduces new terminology, with "negative reverse repetition," and "e-type." He shows that the continued fraction patterns of analogues of simple and general Hurwitz numbers, in certain cases, are of pure "e-type." He points out subtle pattern variations of Hurwitz expansions for cardinality $q = 2$. He also studies special numbers, such as $e/\overline{p}$, where $\overline{p}$ is the degree two prime.

3.1 Results of Euler and Hurwitz

In this section, Dr. Thakur discusses the continued fraction expansions of real numbers first proposed by Euler and Hurwitz. He shows e raised to various fractional powers, and also discusses Hurwitz numbers in the real number setting.

3.1.1 Th(2.1.) $e^{1/n}$ and $e^{2/n}$

First, Dr. Thakur discusses the real number e raised to various fractional powers. He gives the examples of $e^{1/n}$ and $e^{2/n}$.

> ..., Euler showed (the overline in the notation indicates infinite arithmetic progressions), that for n > 1,
>
> $e^{1/n} = \overline{[1, n-1+2in, 1]}_{i=0}^{\infty} = [1,\ n-1,\ 1,\ 1,\ 3n-1,\ 1,\ 1,\ 5n-1,\ 1,\ ...]$
>
> (Thakur, 1996, p. 251)

For example, $e^{1/2}$ to 100 places looks like the following:

[1, 1, 1, 1, 5, 1, 1, 9, 1, 1, 13, 1, 1, 17, 1, 1, 21, 1, 1, 25, 1, 1, 29, 1, 1, 33, 1, 1, 37, 1, 1, 41, 1, 1, 45, 1, 1, 49, 1, 1, 53, 1, 1, 57, 1, 1, 61, 1, 1, 65, 1, 1, 69, 1, 1, 73, 1, 1, 77, 1, 1, 81, 1, 1, 85, 1, 1, 89, 1, 1, 93, 1, 1, 97, 1, 1, 101, 1, 1, 105, 1, 1, 109, 1, 1, 113, 1, 1, 117, 1, 1, 121, 1, 1, 125, 1, 1, 129, 1, 1, ...]

Also, $e^{1/5}$ to 100 places looks like the following:

[1, 4, 1, 1, 14, 1, 1, 24, 1, 1, 34, 1, 1, 44, 1, 1, 54, 1, 1, 64, 1, 1, 74, 1, 1, 84, 1, 1, 94, 1, 1, 104, 1, 1, 114, 1, 1, 124, 1, 1, 134, 1, 1, 144, 1, 1, 154, 1, 1, 164, 1, 1, 174, 1, 1, 184, 1, 1, 194, 1, 1, 204, 1, 1, 214, 1, 1, 224, 1, 1, 234, 1, 1, 244, 1, 1, 254, 1, 1, 264, 1, 1, 274, 1, 1, 284, 1, 1, 294, 1, 1, 304, 1, 1, 314, 1, 1, 324, 1, 1, ...]

and for odd n > 1,

$$e^{2/n} = \overline{[1, (n-1)/2 + 3in, 6n + 12in, (5n-1)/2 + 3in, 1]}_{i=0}^{\infty}$$

(Thakur, 1996, p. 251)

$e^{2/5}$ to 100 places looks like the following:

[1, 2, 30, 12, 1, 1, 17, 90, 27, 1, 1, 32, 150, 42, 1, 1, 47, 210, 57, 1, 1, 62, 270, 72, 1, 1, 77, 330, 87, 1, 1, 92, 390, 102, 1, 1, 107, 450, 117, 1, 1, 122, 510, 132, 1, 1, 137, 570, 147, 1, 1, 152, 630, 162, 1, 1, 167, 690, 177, 1, 1, 182, 750, 192, 1, 1, 197, 810, 207, 1, 1, 212, 870, 222, 1, 1, 227, 930, 237, 1, 1, 242, 990, 252, 1, 1, 257, 1050, 267, 1, 1, 272, 1110, 282, 1, 1, 287, 1170, 297, 1, ...]

3.1.2 Th(2.1.1.) $(ae^{2/n} + b)/(ce^{2/n} + d)$

In this subsection, Dr. Thakur states that general Hurwitz numbers have continued fraction expansions with a fixed number of arithmetic progressions. This means that there is a finite, set number of terms under the repeat bar. For instance, with the example of "$e^{1/n} = \overline{[1, n-1+2in, 1]}_{i=0}^{\infty}$," above, there are three terms under the repeat bar. Each of "1," "$n - 1 + 2in$," and "1" is a term.

Additionally, there is no easy, general formula for determining the arithmetic progression of an arbitrary Hurwitz number.

In particular, $(ae^{2/n} + b)/(ce^{2/n} + d)$ for n a positive integer and a, b, c, d $\in \mathbb{Z}$ with $ab - bc \neq 0$ have all CF's whose partial quotients are eventually in a fixed number of arithmetic progressions. But the process to write down the CF is quite involved and there is no easy "formula" in general. (Thakur, 1996, p. 251)

Please see Appendix D.3, p. 77 for examples.

3.1.2.1 Th(2.1.2.) Two Examples Worked Out by Hurwitz

Then, Dr. Thakur describes general Hurwitz numbers in the real number setting. He gives the examples of $2e$ and $\frac{e+1}{3}$.

We give two examples worked out by Hurwitz[H]:

$$2e = [5,\ 2,\ \overline{3, 2i, 3, 1, 2i, 1}]_{i=1}^{\infty}$$

(Thakur, 1996, p. 251)

$2e$ to 100 places looks like the following:

[5, 2, 3, 2, 3, 1, 2, 1, 3, 4, 3, 1, 4, 1, 3, 6, 3, 1, 6, 1, 3, 8, 3, 1, 8, 1, 3, 10, 3, 1, 10, 1, 3, 12, 3, 1, 12, 1, 3, 14, 3, 1, 14, 1, 3, 16, 3, 1, 16, 1, 3, 18, 3, 1, 18, 1, 3, 20, 3, 1, 20, 1, 3, 22, 3, 1, 22, 1, 3, 24, 3, 1, 24, 1, 3, 26, 3, 1, 26, 1, 3, 28, 3, 1, 28, 1, 3, 30, 3, 1, 30, 1, 3, 32, 3, 1, 32, 1, 3, 34, ...]

$\frac{e+1}{3} =$
$[1,\ 4,\ \overline{5, 4i-3, 1, 1, 36i-16, 1, 1, 4i-2, 1, 1, 36i-4, 1, 1, 4i-1, 1, 5, 4i, 1}]_{i=1}^{\infty}$.
(Thakur, 1996, p. 251)

$\frac{e+1}{3}$ = to 150 places looks like the following:

[1, 4, 5, 1, 1, 1, 20, 1, 1, 2, 1, 1, 32, 1, 1, 3, 1, 5, 4, 1, 5, 5, 1, 1, 56, 1, 1, 6, 1, 1, 68, 1, 1, 7, 1, 5, 8, 1, 5, 9, 1, 1, 92, 1, 1, 10, 1, 1, 104, 1, 1, 11, 1, 5, 12, 1, 5, 13, 1, 1, 128, 1, 1, 14, 1, 1, 140, 1, 1, 15, 1, 5, 16, 1, 5, 17, 1, 1, 164, 1, 1, 18, 1, 1, 176, 1, 1, 19, 1, 5, 20, 1, 5, 21, 1, 1, 200, 1, 1, 22, 1, 1, 212, 1, 1, 23, 1, 5, 24, 1, 5, 25, 1, 1, 236, 1, 1, 26, 1, 1, 248, 1, 1, 27, 1, 5, 28, 1, 5, 29, 1, 1, 272, 1, 1, 30, 1, 1, 284, 1, 1, 31, 1, 5, 32, 1, 5, 33, 1, 1, ...]

3.2 The $\mathbb{F}_q[t]$ Case

This section is the main section of the 1996 article. Dr. Thakur begins by recalling Theorem 1 of the 1992 article. He introduces "negative reverse repetition" terminology, and defines "e-type."

He then goes on to state and prove several theorems related to patterns of continued fraction expansions of analogues of general and simple Hurwitz numbers. He points out that, under certain conditions, they are of pure "e-type." He describes subtle pattern variations for $q = 2$. He also talks about special numbers, such as $e/\overline{p}$, where $\overline{p} = t^2 + t + 1$, the degree two prime.

3.2.1 Th(4.1.) Theorem 1

Dr. Thakur begins this section recalling Theorem 1 from his 1992 article, (Thakur, 1992, p. 152). He also points out the immediate consequences.

> THEOREM 1. Define a sequence x_n with $x_1 := [0, z^{-q}[1]]$ and if $x_n = [a_0, a_1, \dots, a_{2^n-1}]$, then setting
>
> $x_{n+1} := [a_0, a_1, \dots, a_{2^n-1}, -z^{-q^n(q-2)}d_{n+1}/d_n^2, -a_{2^n-1}, \dots, -a_1]$,
>
> Then
>
> $$x_n := \sum_{i=1}^{n} \frac{z^{q^i}}{d_i}$$
>
> In particular, $e(z) = z + \lim_{n\to\infty} x_n$ and for $q = 2$,
>
> $$e = e(1) = [1, \underbrace{\underbrace{\underbrace{[1]}, [2], \underbrace{[1]}}, [3], \underbrace{[1], [2], [1]}}, [4], \underbrace{[1], [2], [1], [3], [1], [2], [1]}, [5], \dots]$$
>
> (More explicitly, for $n > 0$ the n-th partial quotient is $t^{2^{u_n}} - t$ with u_n being the exponent of the highest power of 2 dividing $2n$.) (Thakur, 1996, p. 252)

3.2.2 Th(4.2.) Negative Reverse Repetition Terminology

Terminology is now introduced to talk about patterns with "negative reverse repetition." Also, the term "e-type" is defined.

> By a CF μ of pure e-type with the initial block $\overrightarrow{X} = (a_1, ..., a_{k_1})$ and digits w_i, we mean CF described by its suitable truncations μ_i as follows: Let $\mu_1 := [a_0, a_1, ..., a_{k_1}, w_1]$ and if $\mu_i := [a_0, a_1, ..., a_{k_i}, w_i]$ then
>
> $\mu_{i+1} := [a_0, a_1, ..., a_{k_i}, w_i, -a_{k_i}, -a_{k_i-1}, ..., -a_1, w_{i+1}]$
>
> (Thakur, 1996, p. 252)

As an example of this, it was seen in Chapter 2.2.2, p. 12, that for $e(1)$,

$\mu_2 = [0, [1], [2], [1], [3]]$
$\mu_3 = [0, [1], [2], [1], [3], [1], [2], [1], [4]]$

He then makes it more visual:

> Let's make it more visual: For $\overrightarrow{Y} = (y_1, ..., y_k)$, put $\overleftarrow{Y} = (y_k, ..., y_1)$ and $-\overleftarrow{Y} := (-y_k, -y_{k-1}, ..., -y_1)$. Then we have $\mu_1 = [a_0, \overrightarrow{X}, w_1]$ and for $\mu_i = [a_0, \overrightarrow{Y}, w_i]$ we have $\mu_{i+1} = [a_0, \overrightarrow{Y}, w_i, -\overleftarrow{Y}, w_{i+1}]$ so that
>
> $$\mu = [a_0, \overrightarrow{X}, w_1, -\overleftarrow{X}, w_2, \overrightarrow{X}, -w_1, -\overleftarrow{X}, w_3, \overrightarrow{X}, ...]$$
>
> We say that $x = [(x_i)]$ is of e-type if it is equivalent (see 1.1.6) to some μ of pure e-type, i.e., for some n, the tail x_n' is a tail of some CF of pure e-type. (Thakur, 1996, p. 253)

3.2.3 Th(4.3.) e and $e(\alpha/f) - \alpha/f$

In this next subsection, Dr. Thakur states that e and $e(\alpha/f)-\alpha/f$ are of pure "e-type." An example of this is provided.

> Theorem 1 then shows that e and $e(\alpha/f) - \alpha/f$ for $f \in A - \{0\}$ and $\alpha \in \mathbb{F}_q^*$ (see also 5.1, 5.2) are of pure e-type. (Thakur, 1996, p. 253)

For example, let $q = 3$ and $z = 1$. Then $e(\alpha/f) - \alpha/f = e(1) - 1$. Then,

$x_1 := [0,\ 1^{-3}[1]] = [0,\ [1]]$.

$x_2 := [0,\ [1],\ -[2][1],\ -[1]]$.

$x_3 := [0,\ [1],\ -[2][1],\ -[1],\ -[3][2][1]^3,\ [1],\ [2][1],\ -[1]]$

$e(1) - 1 = [0,$

$[1],\ -[2][1],\ -[1],\ -[3][2][1]^3,\ [1],\ [2][1],\ -[1],\ -[4][3][2]^3[1]^9,$

$[1],\ -[2][1],\ -[1],\ [3][2][1]^3,\ [1],\ [2][1],\ -[1],\ -[5][4][3]^3[2]^9[1]^{27},$

$[1],\ -[2][1],\ -[1],\ -[3][2][1]^3,\ [1],\ [2][1],\ -[1],\ [4][3][2]^3[1]^9,$

$[1],\ -[2][1],\ -[1],\ [3][2][1]^3,\ [1],\ [2][1],\ -[1],\ -[6][5][4]^3[3]^9[2]^{27}[1]^{81},\ ...]$.

3.2.4 Th(4.4.) General and Simple Hurwitz Numbers and "e-Type"

In this subsection, Dr. Thakur proves a lemma and two theorems which show that general and simple Huwitz numbers, under certain conditions, are of pure "*e*-type." In conjunction with this, he extablishes that a general Hurwitz number can be transformed to a simple Huwitz number counterpart with Mobïus transformation.

Dr. Thakur begins by stating that the proof of Theorem 1 is based on a calculation taken from the lemma of (Poorten & Shallit, 1992). The lemma of Poorten and Shallit is corroborated and adapted here.

> The proof of Theorem 1 (see also the remark following the proof in [T1]) is based on a calculation abstracted in the following lemma of [PS], [DMP] whose variants already appear in [S1]. (Thakur, 1996, p. 253)

A detailed proof of Theorem 1 can be found in Chapter 2.2.3, p. 13.

3.2.4.1 Th(4.4.) Lemma 1.

Lemma 1 speaks to how one convergent (i.e., truncated continued fraction) progresses to the next step as n goes from 1 to ∞. A similar stepwise progression is seen with Lemma 2.

LEMMA 1. Let $\overrightarrow{X} = (a_1, ..., a_n)$, so that $[a_0, \overrightarrow{X}] = p_n/q_n$. Then, $[a_0, \overrightarrow{X}, y, -\overleftarrow{X}] = p_n/q_n + (-1)^n/yq_n^2$. (Thakur, 1996, p. 253)

3.2.4.2 Th(4.4.) Proof of Lemma 1.

Proof. To prove Lemma 1, The proof found in Poorten/Shallit (Poorten & Shallit, 1992, p. 239) has been adapted as follows:

$$[a_0, \overrightarrow{X}, y - \tfrac{q_{n-1}}{q_n}] \leftrightarrow \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} y - q_{n-1}/q_n & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} yp_n - (p_n q_{n-1} - p_{n-1}q_n)/q_n & p_n \\ yq_n & q_n \end{pmatrix} \leftrightarrow \tfrac{p_n}{q_n} + \tfrac{(-1)^n}{yq_n^2}$$

since $(p_n q_{n-1} - p_{n-1}q_n) = (-1)^{n-1}$;
and, of course, $y - q_{n-1}/q_n = [y, -\overleftarrow{X}]$. □

3.2.4.3 Th(4.4.) Example of Lemma 1.

A specific example of Lemma 1 follows, with the evaluation of $e(1) - 1$ for $q = 3$. This function is more fully expanded in Appendix E.3.5, p. 84.

Note also, the introduction of new "Curly Bracket" notation, related to cardinality $q = 3$, as defined in Appendix C.1 on p. 73.

Let n = 3 and $\overrightarrow{X} = (a_1, a_2, a_3) = (2t + t^3, 2t^2 + t^4 + t^{10} + 2t^{12}, t + 2t^3)$.
$[a_0, \overrightarrow{X}] = \frac{p_3}{q_3} = \frac{1+2t^3+2t^5+2t^7+t^{11}+t^{13}+t^{15}}{t^4+2t^{10}+2t^{12}+t^{18}}$.

If y is then set to $t^5 + 2t^{11} + 2t^{13} + t^{19} + 2t^{31} + t^{37} + t^{39} + 2t^{45}$,

$[a_0, \overrightarrow{X}, y, -\overleftarrow{X}]$

= [0, {2:1, 1:3}, {2:2, 1:4, 1:10, 2:12}, {1:1, 2:3}, {1:5, 2:11, 2:13, 1:19, 2:31, 1:37, 1:39, 2:45}, {2:1, 1:3}, {1:2, 2:4, 2:10, 1:12}, {1:1, 2:3}]

= ({1:0}, {2:9}, {1:12}, {1:14}, {2:15}, {1:16}, {2:17}, {1:18}, {1:20}, {2:21}, {1:22}, {2:23}, {1:24}, {2:25}, {1:26}, {1:28}, {2:29}, {2:31}, {1:35}, {2:36}, {2:37}, {1:38}, {1:40}, {1:41}, {1:42}, {1:43}, {1:44}, {1:46}, {1:47}, {1:48}, {1:49}, {1:50}, {1:51}, {1:52}, {2:54}, {1:55}, {1:57}, {1:62}, {1:63}, {1:64}, {1:66}, {1:68}, {1:70}, {1:72}, {1:74}, {1:76}, {1:78})/({2:13}, {1:31}, {1:37}, {1:39}, {2:55}, {2:57}, {2:63}, {1:81}) (Note, this is a quotient.)

$$= \frac{1+2t^3+2t^5+2t^7+t^{11}+t^{13}+t^{15}}{t^4+2t^{10}+2t^{12}+t^{18}}$$
$$+ \frac{-1}{(t^5+2t^{11}+2t^{13}+t^{19}+2t^{31}+t^{37}+t^{39}+2t^{45})(t^4+2t^{10}+2t^{12}+t^{18})^2}$$
$$= \frac{p_3}{q_3} + \frac{(-1)^3}{yq_3^2} = \frac{z^q}{D_1} + \frac{z^{q^2}}{D_2} + \frac{z^{q^3}}{D_3} = \frac{1^3}{D_1} + \frac{1^{3^2}}{D_2} + \frac{1^{3^3}}{D_3} = \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3}$$

So, Lemma 1 is demonstrated for the example of $e(1) - 1$ with $q = 3$ for three terms with Mathematica 5.2.

3.2.4.4 Th(4.4.) Theorem 2.

Theorem 2 is the main result of this section. As stated, it relates to cardinality $q > 2$.

The Main Result when $q > 2$

THEOREM 2. Let $q > 2$, a, b, c, d, $f \in A$, $f \neq 0$. If $ad - bc \neq 0$, then $(ae(1/f) + b)(ce(1/f) + d)$ is of e-type. If a, b, d, $\neq 0$, then $M := (a/b)e(1/f) + (c/d)$ is of pure e-type. (Thakur, 1996, p. 253)

3.2.4.5 Th(4.4.) Proof of Theorem 2.

Here, Dr. Thakur proves Theorem 2. He begins by proving the second claim. By this, he means the second inline statement of the theorem, related to analogues of simple Hurwitz numbers. This is, if a, b, $d \neq 0$, then $M := (a/b)e(1/f) + (c/d)$ is of pure e-type.

Proof. We first prove the second claim. We have

$$M = \{(\tfrac{c}{d} + \frac{1}{bf/a} + \cdots + \frac{1}{bf^{q^{n-1}}d_{n-1}/a}) + \frac{1}{bf^{q^n}d_n/a}\}$$
$$+ \frac{1}{bf^{q^{n+1}}d_{n+1}/a} \cdots$$

Let $(n-1)$ be a positive integer greater than the degrees of a, b, and d. Let the quantity (call it η) in the curly brackets $\{\ \}$ in the displayed equation have CF $[a_0, \overrightarrow{X}]$. (Thakur, 1996, p. 253)

Note that the expansion for M above was derived as follows:

$$M = (a/b)\ e(1/f) + c/d = (a/b)\ (\textstyle\sum_{i=0}^{n} \frac{(1/f)^{q^i}}{d_i}) + c/d$$
$$= c/d + (a/b)(1/f) + \frac{(a/b)(1/f)^q}{d_1} + \frac{(a/b)(1/f)^{q^2}}{d_2} + \frac{(a/b)(1/f)^{q^3}}{d_3} + \cdots$$
$$= \{(\tfrac{c}{d} + \tfrac{1}{bf/a} + \cdots + \tfrac{1}{bf^{q^{n-1}}d_{n-1}/a}) + \tfrac{1}{bf^{q^n}d_n/a}\} + \tfrac{1}{bf^{q^{n+1}}d_{n+1}/a} \cdots$$

A specific example of this is as follows. With the development of the simple Hurwitz expansion of $(\frac{t+2}{2t})e(\frac{1}{t}) + \frac{t}{t+1}$, (FHE3-1), in Appendix E.5, p. 92, $q = 3$, and $(n-1) = 2$. Also, $a = t+2$, $b = 2t$, $c =$ t, $d = t+1$, $\theta = 1$, and $f = t$.

A specific example of M is found to be:

$$M = \{(\tfrac{c}{d} + \tfrac{1}{bf/a} + \tfrac{1}{bf^3d_1/a} + \tfrac{1}{bf^9d_2/a}) + \tfrac{1}{bf^{27}d_3/a}\} + \tfrac{1}{bf^{81}d_4/a} \cdots$$
$$= \{(\tfrac{t}{t+1} + \tfrac{1}{2t^2/(t+2)} + \tfrac{1}{2t^4(t^3-t)/(t+2)} + \tfrac{1}{2t^{10}(t^9-t)(t^3-t)^3/(t+2)})$$
$$+ \tfrac{1}{2t^{28}(t^{27}-t)(t^9-t)^3(t^3-t)^9/(t+2)}\} + \tfrac{1}{2t^{82}(t^{81}-t)(t^{27}-t)^3(t^9-t)^9(t^3-t)^{27}/(t+2)} \cdots$$

Now refer to "Colon" notation defined in Appendix C.2 on p. 74. η in the curly brackets { } in the above specific equation reduced modulo 3 has a simple CF expansion of:

$[a_0, \overrightarrow{X}]$ = [1, {1:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {1:0, 2:1}, {1:0, 2:1},
{2:4, 2:5, 2:6, 2:7, 2:8, 2:9, 1:12, 1:13, 1:14, 1:15, 1:16, 1:17},
{2:0, 1:1}, {2:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {2:0, 2:1},
{1:13, 2:14, 2:19, 1:20, 2:21, 1:22, 1:27, 2:28, 2:39, 1:40, 1:45, 2:46, 1:47, 2:48, 2:53, 1:54},
{1:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {1:0, 2:1}, {1:0, 2:1},
{1:4, 1:5, 1:6, 1:7, 1:8, 1:9, 2:12, 2:13, 2:14, 2:15, 2:16, 2:17},
{2:0, 1:1}, {2:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {2:0, 2:1}]

Note that the length of the initial block $\overrightarrow{X}$ is 23. This length will be defined as k_1 in the next claim.

3.2.4.6 Th(4.4.) Claim 1 - Theorem 2

Next, Dr. Thakur states his first explicit claim related to Theorem 2, which will be referred to as Claim 1 for Theorem 2. He uses this and a second explicit claim to prove the theorem.

> CLAIM. The CF of M is of pure e-type with the initial block $\overrightarrow{X}$ and digits $w_i := (-1)^{k_i} a(f^{q^{n+i-1}} d_{n+i-1})^{q-2}[n+i]/b$, for $i = 1$ to ∞ (note that these are nonconstant integers, by 3.1.2, since $q > 2$), where $k_i = 1$ for $i > 1$ and k_1 is the length of the initial block. (Thakur, 1996, p. 253)

Note that with (FHE3-1), in Appendix E.5, p. 92, $q = 3$, $(n$ - $1) = 2$, and $k_1 = 23$.

A specific example of w_i is found to be:

$w_1 := (-1)^{k_i} a(f^{q^{n+i-1}} d_{n+i-1})^{q-2}[n+i]/b = (-1)^{23} a(f^{27} d_3)[4]/b$
$= -\frac{(t+2)(t^{27})((t^{27}-t)(t^9-t)^3(t^3-t)^9)(t^{81}-t)}{2t}$
= {2:40, 1:41, 1:58, 2:59, 1:64, 2:65, 1:66, 2:67, 2:82, 1:83, 2:84, 1:85, 2:90, 1:91, 1:108, 2:109, 1:120, 2:121, 2:138, 1:139, 2:144, 1:145, 2:146, 1:147, 1:162, 2:163, 1:164, 2:165, 1:170, 2:171, 2:188, 1:189} (Note, this is a polynomial in t.)

$w_2 := (-1)^1 a(f^{81} d_4)[5]/b = -\frac{(t+2)(t^{81})((t^{81}-t)(t^{27}-t)^3(t^9-t)^9(t^3-t)^{27})(t^{243}-t)}{2t}$
= {1:121, 2:122, 2:175, 1:176, 2:193, 1:194, 2:199, 1:200, 2:201, 1:202, 1:247, 2:248, 1:253, 2:254, 1:255, 2:256, 1:271, 2:272, 1:273, 2:274, 1:279, 2:280, 2:325, 1:326, 2:327, 1:328, 2:333, 1:334, 2:351, 1:352, 2:363, 1:364, 1:405, 2:406, 1:417, 2:418, 1:435, 2:436, 1:441, 2:442, 1:443, 2:444, 2:489, 1:490, 2:495, 1:496, 2:497, 1:498, 2:513, 1:514, 2:515, 1:516, 2:521, 1:522, 1:567, 2:568, 1:569, 2:570, 1:575, 2:576, 1:593, 2:594, 2:647, 1:648} (Note, this is a polynomial in t.)

Note that this agrees exactly with what was calculated for (FHE3-1) using the "Rational Function to Simple CF Expansion Algorithm," described in Appendix 3, p. 80.

> *Proof of the Claim.* By 3.1.2, our choice of n implies that the quantity in the round brackets ( ) in the displayed equation can be written with common (integral) denominator $bf^{q^{n-1}} d_{n-1}/a$, but it may not be the reduced denominator. (Thakur, 1996, p. 254)

In the example above,

$$\begin{aligned} bf^{q^{n-1}}d_{n-1}/a &= bf^9 d_2/a = 2t^{10}(t^9-t)(t^3-t)^3/(t+2) \\ &= t^{14}+t^{15}+t^{16}+t^{17}+t^{18}+t^{19}+2t^{22}+2t^{23}+2t^{24}+2t^{25}+2t^{26}+2t^{27} \\ &= \{1{:}14,\ 1{:}15,\ 1{:}16,\ 1{:}17,\ 1{:}18,\ 1{:}19,\ 2{:}22,\ 2{:}23,\ 2{:}24,\ 2{:}25,\ 2{:}26,\ 2{:}27\} \end{aligned}$$

> On the other hand, since by our choice of n every prime dividing $bf^{q^n}d_n/a$ divides d_n or f, it will not divide the numerator of η written with the denominator $bf^{q^n}d_n/a$ showing that it is in fact the reduced denominator of η. (Thakur, 1996, p. 254)

As an example, look at

$$\begin{aligned} \eta &= \{(\tfrac{c}{d} + \tfrac{1}{bf/a} + \tfrac{1}{bf^3 d_1/a} + \tfrac{1}{bf^9 d_2/a}) + \tfrac{1}{bf^{27}d_3/a}\} \\ &= \{(\tfrac{t}{t+1} + \tfrac{1}{2t^2/(t+2)} + \tfrac{1}{2t^4(t^3-t)/(t+2)} + \tfrac{1}{2t^{10}(t^9-t)(t^3-t)^3/(t+2)}) \\ &\quad + \tfrac{1}{2t^{28}(t^{27}-t)(t^9-t)^3(t^3-t)^9/(t+2)}\}. \end{aligned}$$

Simplify and reduce this modulo 3 to get:

Numerator of η = {2:0, 1:27, 1:33, 1:35, 2:36, 2:38, 2:39, 2:40, 1:41, 1:43, 1:47, 1:49, 2:53, 1:54, 1:55, 1:56, 2:57, 1:58, 2:59, 1:60, 2:61, 2:62, 2:63, 2:64, 1:65, 1:66, 2:67, 2:69, 2:73, 2:75, 2:78, 1:80, 1:82, 1:83, 2:84, 2:86, 2:88, 1:89, 2:90, 1:104, 1:106, 2:107, 1:108}

Denominator of η = {1:41, 1:42, 1:43, 1:44, 1:45, 1:46, 1:47, 1:48, 1:49, 1:50, 1:51, 1:52, 1:53, 1:54, 1:55, 1:56, 1:57, 1:58, 2:65, 2:66, 1:67, 1:68, 1:69, 1:70, 1:71, 1:72, 1:73, 1:74, 1:75, 1:76, 1:77, 1:78, 1:79, 1:80, 1:81, 1:82, 2:83, 2:84, 1:91, 1:92, 1:93, 1:94, 1:95, 1:96, 1:97, 1:98, 1:99, 1:100, 1:101, 1:102, 1:103, 1:104, 1:105, 1:106, 1:107, 1:108}

The prime, $t+1$, does not divide evenly into the numerator, but it does divide evenly into the denominator.

The polynomial quotient of $\frac{\textit{Denominator of }\eta}{t+1}$ = {1:41, 1:43, 1:45, 1:47, 1:49, 1:51, 1:53, 1:55, 1:57, 2:65, 1:67, 1:69, 1:71, 1:73, 1:75, 1:77, 1:79, 1:81, 2:83, 1:91, 1:93, 1:95, 1:97, 1:99, 1:101, 1:103, 1:105, 1:107}

> Since $bf^{q^m}d_m/a = (bf^{q^{m-1}}d_{m-1}/a)^2 a(f^{q^{m-1}}d_{m-1})^{q-2}[m]/b$, Lemma 1 finishes the proof of the claim and hence of the second claim in the Theorem by induction on m. (Thakur, 1996, p. 254)

With the example, letting $m = 3$, the following will be shown:

$$bf^{27}d_3/a = (bf^9d_2/a)^2a(f^9d_2)[3]/b$$

To see this, note that:

$$bf^{27}d_3/a = \frac{2t^{28}(t^{27}-t)(t^9-t)^3(t^3-t)^9}{t+2} =$$
$$= \frac{2t^{19}(t^9-t)^2(t^3-t)^6t^9(t^9-t)(t^3-t)^3(t^{27}-t)}{t+2}$$
$$= (\frac{2t^{19}(t^9-t)^2(t^3-t)^6}{(t+2)^2})(t+2)t^9(t^9-t)(t^3-t)^3(t^{27}-t)$$
$$= \frac{(\frac{4t^{20}(t^9-t)^2(t^3-t)^6}{(t+2)^2})(t+2)t^9(t^9-t)(t^3-t)^3(t^{27}-t)}{2t}$$
$$= \frac{(\frac{2t^{10}(t^9-t)(t^3-t)^3}{t+2})^2(t+2)t^9(t^9-t)(t^3-t)^3(t^{27}-t)}{2t}$$
$$= (bf^9d_2/a)^2a(f^9d_2)[3]/b$$

Observe the following analogue of 1.1.6 in Chapter 2.2.1, p. 10. This is found in Baum/Sweet (Baum & Sweet, 1976).

Remark. It is in fact true that power series f and f^* are equivalent (i.e., $f^* = (g_1f + h_1)/(g_2f + h_2)$ with $g_1h_2 + g_2h_1 = 1$ and $g_1, g_2, h_1, h_2 \in F[x]$) if and only if there exist m and n such that $a_{n+i}(f) = a_{m+i}(f^*)$ for all $i \geqq 0$. (Baum & Sweet, 1976, p. 601)

The above analogue/remark will be used to establish the following claim, and thereby deduce the first claim in the Theorem.

3.2.4.7 Th(4.4.) Claim 2 - Theorem 2:

Mobïus Transformation which "keeps the tails the same."

Next, Dr. Thakur states the second explicit claim related to Theorem 2. This will be referred to as Claim 2 for Theorem 2, and is the second explicit claim used in the proof of the theorem.

CLAIM. Let a, b, c, $d \in A$, with $D := ad - bc \neq 0$ be given. Then, there are a', b', c', $d' \in A$, with $a'd' - b'c' = 1$ and r_1, $r_2 \in K$ satisfying

$$\frac{ax+b}{cx+d} = \frac{a'(r_1x+r_2)+b'}{c'(r_1x+r_2)+d'} \in K(x)$$

(Thakur, 1996, p. 254)

Important Result of Claim 2 *for Theorem* 2.

Claim 2 for Theorem 2 shows how to perform a Mobïus transformation which "keeps the tails the same" of a general Hurwitz number to a simple Hurwitz number. The resulting general and simple Hurwitz numbers might not be the same, but their tails will be the same.

This means that after having found r_1 and r_2, as specified above, the tails of $\frac{ae(1/f)+b}{ce(1/f)+d}$ and $\frac{a''}{b''}e(1/f) + \frac{c''}{d''}$ will be the same, where $\frac{a''}{b''} = r_1$ and $\frac{c''}{d''} = r_2$. They will be identical with the possible exceptions of varying in sign and with starting at different partial quotients for each expansion.

Proof of Claim 2 *for Theorem* 2.

First, look at the more complicated case where all of a, b, c, $d \neq 0$.

> Let g be the greatest common divisor of a and c. Then $a' := a/g \in A$ and $c' := c/g \in A$ are relatively prime and hence there are b' and d' in A such that $a'd' - b'c' = 1$. (Thakur, 1996, p. 254)

See Theorem 2.12, p. 62, of Dence/Dence (Dence & Dence, 1999). (In consulting with Dr. Thakur in July 2006, it was confirmed that the expression $r_2 := -(bg + b'D)/(a'D)$ should be $r_2 := (bg - b'D)/(a'D)$.)

> With $r_2 := (bg - b'D)/(a'D) \in K$, we have $b/d = (a'r_2 + b')/(c'r_2 + d')$. Hence we can solve for $r_1, t \in K$ such that $g = r_1t$, $b = (a'r_2 + b')t$ and $d = (c'r_2 + d')t$. But this is equivalent to the displayed equation in the claim. This finishes the proof of the claim and hence of the Theorem. (Thakur, 1996, p. 254)

Observe that defining $r_2 := (bg-b'D)/(a'D)$ implies $b/d = (a'r_2+b')/(c'r_2+d')$. It is given that $a'd' - b'c' = 1$. First, multiply both sides by g.

Since $a' := a/g$ and $c' := c/g$:

$ad' - b'c = g$

$ad' - b'c - g = 0$

$b'c - ad' + g = 0$

Then,

$$\frac{b}{d} = \frac{abg}{adg} = \frac{abg}{ad(ad'-b'c)+bc(b'c-ad'+g)} = \frac{b'+\frac{bg-b'(ad-bc)}{ad-bc}}{d'+\frac{c(bg-b'(ad-bc))}{a(ad-bc)}}$$

$$= \frac{b'+\frac{bg-b'D}{D}}{d'+\frac{c(bg-b'D)}{aD}} = \frac{b'+a'(\frac{bg-b'D}{a'D})}{d'+c'(\frac{bg-b'D}{a'D})} = \frac{a'r_2+b'}{c'r_2+d'} \ \square$$

Secondly, look at the easier case when one of a, b, c, d is zero. Dr. Thakur assisted with this analysis by e-mail, July 18, 2006. For example, let $d = 0$.

We are trying to show:

$$\frac{ax+b}{cx} = \frac{a'(r_1x+r_2)+b'}{c'(r_1x+r_2)+d'} \in K(x)$$

(Thakur, 1996, p. 254)

Define $r_1 = -\frac{g^2}{bc}$ and $r_2 = -\frac{d'}{c'}$. Once again, $ad' - b'c = g$.

$$\frac{ax+b}{cx} = \frac{agx+bg}{cgx} = \frac{agx+b(ad'-b'c)}{cgx} = \frac{a(bd'+gx)-bb'c}{cgx}$$

$$= \frac{-a(bd'+gx)+bb'c}{-cgx} = \frac{-\frac{ab(bd'+gx)}{bc}+bb'}{-gx} = \frac{-\frac{a(bd'+gx)}{bc}+b'}{-\frac{g}{b}x}$$

$$= \frac{-\frac{ag}{bc}x-\frac{ad'}{c}+b'}{-\frac{g}{b}x} = \frac{-\frac{ag}{bc}x-\frac{ad'}{c}+b'}{-\frac{g}{b}x-d'+d'} = \frac{\frac{a}{g}(-\frac{g^2}{bc}x-\frac{d'g}{c})+b'}{\frac{c}{g}(-\frac{g^2}{bc}x-\frac{d'g}{c})+d'}$$

$$= \frac{a'(-\frac{g^2}{bc}x-\frac{d'}{c'})+b'}{c'(-\frac{g^2}{bc}x-\frac{d'}{c'})+d'} = \frac{a'(r_1x+r_2)+b'}{c'(r_1x+r_2)+d'} \ \square$$

Two cases related to Mobïus transformation and Claim 2 for Theorem 2, as found on page 34 have been proved. The first was the case where all of a, b, c, $d \neq 0$, and the second was the case where $d = 0$.

Example 1 *of Claim* 2 *for Theorem* 2: *All of* a, b, c, $d \neq 0$.

An example of *Claim* 2 for the general Hurwitz number $\frac{(t+2)e(1/t)+2t}{2e(1/t)+(t+1)}$ follows.

Note that for this example, $q = 3$, $(n-1) = 2$, $a = t+2$, $b = 2t$, $c = 2$, $d = t+1$, $\theta = 1$, and $f = t$.

This is a case where all of a, b, c, $d \neq 0$. r_1, r_2 need to be found, for which:

$\frac{ax+b}{cx+d} = \frac{a'(r_1x+r_2)+b'}{c'(r_1x+r_2)+d'} \in K(x)$

$g = \gcd(t+2, 2) = 2$

$a' = \frac{a}{g} = 2t+1$, $c' = \frac{c}{g} = 1$

Defining $b' := 2t$, $d' := 1$ gives one pair of values which satisfies $a'd' - b'c' = 1$.

$a'd' - b'c' = (2t+1) - 2t = 1$ (modulo 3).

Remember that $D := ad - bc = t^2 + 2t + 2$.

$r_2 := (bg - b'D)/(a'D) = \frac{2t(2)-2tD}{(2t+1)D} = \frac{2t^2}{t^2+2t+2}$

$\frac{ax+b}{cx+d} = \frac{(t+2)x+2t}{2x+(t+1)} = \frac{a'(r_1x+r_2)+b'}{c'(r_1x+r_2)+d'} = \frac{(2t+1)(r_1x+\frac{2t^2}{t^2+2t+2})+2t}{(r_1x+\frac{2t^2}{t^2+2t+2})+1}$

or

$\frac{(t+2)x+2t}{2x+(t+1)} = \frac{(2t+1)(r_1x+\frac{2t^2}{t^2+2t+2})+2t}{(r_1x+\frac{2t^2}{t^2+2t+2})+1}$

Solving the above equation for r_1 and reducing modulo 3 gives:

$r_1 = \frac{1}{t^2+2t+2}$

So, one pair of values, r_1, r_2 which satisfies *Claim* 2 is

$r_1 = \frac{1}{t^2+2t+2}$, and $r_2 = \frac{2t^2}{t^2+2t+2}$.

Then,

$\frac{ax+b}{cx+d} = \frac{(t+2)x+2t}{2x+(t+1)} = \frac{a'(r_1x+r_2)+b'}{c'(r_1x+r_2)+d'} = \frac{(2t+1)((\frac{1}{t^2+2t+2})x+\frac{2t^2}{t^2+2t+2})+2t}{((\frac{1}{t^2+2t+2})x+\frac{2t^2}{t^2+2t+2})+1}$

So, importantly, with Mobïus transformation, the general Hurwitz number is converted to its simple Hurwitz form as:

$\frac{(t+2)e(1/t)+2t}{2e(1/t)+(t+1)} \sim$ (has the same tail as) $\frac{1}{t^2+2t+2}e(1/t) + \frac{2t^2}{t^2+2t+2}$.

Observe the first 33 partial quotients of each expression above. Note that the tails match, except for sign, after the fourth and fifth partial quotients, respectively.

$\frac{(t+2)e(1/t)+2t}{2e(1/t)+(t+1)} =$ [2, {1:0, 2:1}, {1:0, 2:1}, {1:0, 1:1, 1:2}, {1:0, 1:1, 1:2}, {2:0, 2:1}, {1:5, 2:6, 1:7, 1:8, 1:10, 2:11, 2:12, 2:13}, {1:0, 1:1}, {2:0, 2:1, 2:2}, {2:0, 2:1, 2:2}, {2:0, 1:1}, {2:0, 1:1}, {2:0, 2:1},

{1:14, 2:15, 2:16, 2:18, 1:19, 1:21, 1:22, 1:24, 2:25, 2:26, 2:40, 1:41, 1:42, 1:44, 2:45, 2:47, 2:48, 2:50, 1:51, 1:52},

{1:0, 1:1}, {1:0, 2:1}, {1:0, 2:1}, {1:0, 1:1, 1:2}, {1:0, 1:1, 1:2}, {2:0, 2:1},

{2:5, 1:6, 2:7, 2:8, 2:10, 1:11, 1:12, 1:13},

{1:0, 1:1}, {2:0, 2:1, 2:2}, {2:0, 2:1, 2:2}, {2:0, 1:1}, {2:0, 1:1}, {2:0, 2:1},

{2:41, 1:42, 1:43, 1:45, 2:46, 2:47, 2:49, 1:50, 1:51, 1:53, 2:54, 2:55, 2:57, 1:58, 2:59, 2:60, 2:62, 1:63, 1:64, 1:65, 2:67, 1:68, 1:69, 1:71, 2:72, 2:73, 2:75, 1:76, 1:77, 1:79, 2:80, 2:81, 1:83, 2:84, 1:85, 1:86, 1:88, 2:89, 2:90, 2:91, 2:93, 1:94, 1:95, 1:97, 2:98, 2:99, 2:101, 1:102, 1:103, 1:105, 2:106, 2:107, 1:121, 2:122, 2:123, 2:125, 1:126, 1:127, 1:129, 2:130, 2:131, 2:133, 1:134, 1:135, 1:137, 2:138, 1:139, 1:140, 1:142, 2:143, 2:144, 2:145, 1:147, 2:148, 2:149, 2:151, 1:152, 1:153, 1:155, 2:156, 2:157, 2:159, 1:160, 1:161, 2:163, 1:164, 2:165, 2:166, 2:168, 1:169, 1:170, 1:171, 1:173, 2:174, 2:175, 2:177, 1:178, 1:179, 1:181, 2:182, 2:183, 2:185, 1:186, 1:187},

{1:0, 1:1}, {1:0, 2:1}, {1:0, 2:1}, {1:0, 1:1, 1:2}, {1:0, 1:1, 1:2}]

$\frac{1}{t^2+2t+2}e(1/t) + \frac{2t^2}{t^2+2t+2}$ = [2, {2:0, 2:1}, {2:0, 1:1}, {2:0, 1:1}, {2:0, 2:1, 2:2}, {2:0, 2:1, 2:2}, {1:0, 1:1},

{2:5, 1:6, 2:7, 2:8, 2:10, 1:11, 1:12, 1:13},

{2:0, 2:1}, {1:0, 1:1, 1:2}, {1:0, 1:1, 1:2}, {1:0, 2:1}, {1:0, 2:1}, {1:0, 1:1},

{2:14, 1:15, 1:16, 1:18, 2:19, 2:21, 2:22, 2:24, 1:25, 1:26, 1:40, 2:41, 2:42, 2:44, 1:45, 1:47, 1:48, 1:50, 2:51, 2:52},

{2:0, 2:1}, {2:0, 1:1}, {2:0, 1:1}, {2:0, 2:1, 2:2}, {2:0, 2:1, 2:2}, {1:0, 1:1},

{1:5, 2:6, 1:7, 1:8, 1:10, 2:11, 2:12, 2:13},

{2:0, 2:1}, {1:0, 1:1, 1:2}, {1:0, 1:1, 1:2}, {1:0, 2:1}, {1:0, 2:1}, {1:0, 1:1},

{1:41, 2:42, 2:43, 2:45, 1:46, 1:47, 1:49, 2:50, 2:51, 2:53, 1:54, 1:55, 1:57, 2:58, 1:59, 1:60, 1:62, 2:63, 2:64, 2:65, 1:67, 2:68, 2:69, 2:71, 1:72, 1:73, 1:75, 2:76, 2:77, 2:79, 1:80, 1:81, 2:83, 1:84, 2:85, 2:86, 2:88, 1:89, 1:90, 1:91, 1:93, 2:94, 2:95, 2:97, 1:98, 1:99, 1:101, 2:102, 2:103, 2:105, 1:106, 1:107, 2:121, 1:122, 1:123, 1:125, 2:126, 2:127, 2:129, 1:130, 1:131, 1:133, 2:134, 2:135, 2:137, 1:138, 2:139, 2:140, 2:142, 1:143, 1:144, 1:145, 2:147, 1:148, 1:149, 1:151, 2:152, 2:153, 2:155, 1:156, 1:157, 1:159, 2:160, 2:161, 1:163, 2:164, 1:165, 1:166, 1:168, 2:169, 2:170, 2:171, 2:173, 1:174, 1:175, 1:177, 2:178, 2:179, 2:181, 1:182, 1:183, 1:185, 2:186, 2:187},

{2:0, 2:1}, {2:0, 1:1}, {2:0, 1:1}, {2:0, 2:1, 2:2}]

Example 2 *of Claim* 2 *for Theorem* 2: $d = 0$.

An example of *Claim* 2 for the general Hurwitz number $\frac{(t+2)e(1/t)+2t}{2e(1/t)}$ follows.

Note that for this example, $q = 3$, $(n-1) = 2$, $a = t+2$, $b = 2t$, $c = 2$, $d = 0$, $\theta = 1$, and $f = t$.

This is a case where all of a, b, $c \neq 0$, and $d = 0$. r_1, r_2 need to be found, for which:

$\frac{ax+b}{cx+d} = \frac{a'(r_1x+r_2)+b'}{c'(r_1x+r_2)+d'} \in K(x)$

$g = \gcd(t+2, 2) = 2$

$a' = \frac{a}{g} = 2t+1$, $c' = \frac{c}{g} = 1$

Defining $b' := 2t$, $d' := 1$ gives one pair of values which satisfies $a'd' - b'c' = 1$.

$a'd' - b'c' = (2t+1) - 2t = 1$ (modulo 3).

Define $r_1 = -\frac{g^2}{bc}$ and $r_2 = -\frac{d'}{c'}$.

$r_1 = -\frac{2^2}{2t(2)} = \frac{2}{t}$.

$r_2 = -\frac{1}{1} = 2$.

So, with Mobïus transformation, the general Hurwitz number is converted to its simple Hurwitz form as:

$\frac{(t+2)e(1/t)+2t}{2e(1/t)} \sim$ (has the same tail as) $\frac{2}{t}e(1/t) + 2$.

Observe the first 33 partial quotients of each expression above. Note that the tails match, except for sign, after the second and third partial quotients, respectively.

$\frac{(t+2)e(1/t)+2t}{2e(1/t)}$ = [{1:0, 2:1, 1:2}, {1:1, 2:3}, {2:2}, {2:4, 1:6, 1:12, 2:14}, {1:2}, {2:1, 1:3}, {2:2},

{1:13, 2:19, 2:21, 1:27, 2:39, 1:45, 1:47, 2:53},

{1:2}, {1:1, 2:3}, {2:2}, {1:4, 2:6, 2:12, 1:14}, {1:2}, {2:1, 1:3}, {2:2},

{2:40, 1:58, 1:64, 1:66, 2:82, 2:84, 2:90, 1:108, 1:120, 2:138, 2:144, 2:146, 1:162, 1:164, 1:170, 2:188},

{1:2}, {1:1, 2:3}, {2:2}, {2:4, 1:6, 1:12, 2:14}, {1:2}, {2:1, 1:3}, {2:2},

{2:13, 1:19, 1:21, 2:27, 1:39, 2:45, 2:47, 1:53},

{1:2}, {1:1, 2:3}, {2:2}, {1:4, 2:6, 2:12, 1:14}, {1:2}, {2:1, 1:3}, {2:2},

{1:121, 2:175, 2:193, 2:199, 2:201, 1:247, 1:253, 1:255, 1:271, 1:273, 1:279, 2:325, 2:327, 2:333, 2:351, 2:363, 1:405, 1:417, 1:435, 1:441, 1:443, 2:489, 2:495, 2:497, 2:513, 2:515, 2:521, 1:567, 1:569, 1:575, 1:593, 2:647},

{1:2}]

$\frac{2}{t}e(1/t) + 2 =$ [2, {2:2}, {2:1, 1:3}, {1:2}, {1:4, 2:6, 2:12, 1:14}, {2:2}, {1:1, 2:3}, {1:2},

{2:13, 1:19, 1:21, 2:27, 1:39, 2:45, 2:47, 1:53},

{2:2}, {2:1, 1:3}, {1:2}, {2:4, 1:6, 1:12, 2:14}, {2:2}, {1:1, 2:3}, {1:2},

{1:40, 2:58, 2:64, 2:66, 1:82, 1:84, 1:90, 2:108, 2:120, 1:138, 1:144, 1:146, 2:162, 2:164, 2:170, 1:188},

{2:2}, {2:1, 1:3}, {1:2}, {1:4, 2:6, 2:12, 1:14}, {2:2}, {1:1, 2:3}, {1:2},

{1:13, 2:19, 2:21, 1:27, 2:39, 1:45, 1:47, 2:53},

{2:2}, {2:1, 1:3}, {1:2}, {2:4, 1:6, 1:12, 2:14}, {2:2}, {1:1, 2:3}, {1:2},

{2:121, 1:175, 1:193, 1:199, 1:201, 2:247, 2:253, 2:255, 2:271, 2:273, 2:279, 1:325, 1:327, 1:333, 1:351, 1:363, 2:405, 2:417, 2:435, 2:441, 2:443, 1:489, 1:495, 1:497, 1:513, 1:515, 1:521, 2:567, 2:569, 2:575, 2:593, 1:647}]

3.2.5 Th(4.5.) Remark.

Dr. Thakur states here that Theorem 2 is an effective and finite procedure for finding the pattern for the analogues of Hurwitz numbers. Also, the process can sometimes be simplified by using smaller "building blocks."

> *Remark.* Notice that we have given, together with 1.1.6, an effective (finite) procedure to determine the pattern for analogues of Hurwitz numbers. Also, note that sometimes we can use a smaller size "building block" $\overrightarrow{X}$ by taking a smaller n than prescribed above (and keeping the rest of the recipe in the claim the same). (Thakur, 1996, p. 254)

3.2.6 Th(4.6.) Continued Fraction Expansion Subtleties with Cardinality $q = 2$

In this subsection, Dr. Thakur introduces the subtle pattern variations which occur with cardinality $q = 2$. He points out, that with cardinality 2, "negative reverse repetition," "reverse repetition," or just "repetition" are synonymous. Additionally, with a theorem, he proves that continued fraction expansions, under certain conditions, are comprised of repeating building blocks.

> When $q = 2$, the situation is more subtle. Now $-\overleftarrow{X} = \overleftarrow{X}$ and for $q = 2$ the expansion of e given in the Theorem 1 can be equally interpreted as negative reverse repetition or reverse repetition or just repetition. But these interpretations will lead to distinct generalization, as we will see shortly. (Thakur, 1996, p. 254)

3.2.6.1 Th(4.6.) Theorem 3.

With Theorem 3, Dr. Thakur digs more deeply into the properties of patterns for cardinality $q = 2$. Much of the focus of this theorem is with respect to properties of polynomial b.

> THEOREM 3. Let $q = 2$, and let a, b, c, d, $f \in A$, b, d, $f \neq 0$. Let $M := (a/b)e(1/f) + c/d$. If b divides $t^2 + t$, then M is of pure e-type. If, in addition t divides b, (and does not divide a) then $N := (a/b)e(t/f) + c/d$ is also of pure e-type. If b is a square-free polynomial, then for infinitely many n, the CF for M is of the form $[a_0, \overrightarrow{X}, a_n, -\overleftarrow{X}, ...]$ with $\overrightarrow{X} = (a_1, ..., a_{n-1})$. (Thakur, 1996, p. 254)

3.2.6.2 Th(4.6.) Proof of Theorem 3.

Dr. Thakur proceeds the same way with the proof of this Theorem as he did with the proof of Theorem 2. He uses the definition of $[i]$ to show that b divides $[n + 1]$. He relies here, as he did with the proof of Claim 2 for Theorem 2, on the fact that key quantities are integral.

> *Proof*. We proceed exactly as in the case $q > 2$. The fact that b divides d_n^{q-2} in that case may not carry over, but b still divides $[n + 1]$, by 3.1.1 and we get the proof of the first claim. (Thakur, 1996, p. 255)

(The first claim of Theorem 3 is: Let $M := (a/b)e(1/f) + c/d$. If b divides $t^2 + t$, then M is of pure e-type.)

Specifically, since $[n+1] := t^{2^{n+1}} - t$ is the product of monic irreducible elements of A of degree dividing $n+1$, and $n-1 >$ degree of b, it is clear that $b|[n+1]$.

With the example of Theorem 3, below, Chapter 3.2.6.3, p. 44, it is seen that $a = t+1$, $b = t^2+t$, $c = 1$, $d = t$, and $f = t+1$. The maximum degree of a, b, or d in the example is 2, so $(n-1) = 3$ can be chosen, or $n = 4$.

In this case, b does divide d_4, and the polynomial quotient is:

$\frac{d_4}{t^2+t} = \{14, 15, 16, 17, 18, 19, 20, 21, 26, 27, 29, 30, 31, 32, 33, 36, 40, 43, 44, 45, 46, 47, 49, 50, 55, 56, 57, 58, 59, 60, 61, 62\}$.

b also divides $[n+1] = [5]$ and the polynomial quotient is:

$\frac{[5]}{t^2+t} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\}$.

> For the second claim, note that for large n, under the given condition both the terms in ( ) in the equation
>
> $$bf^{2^{n+1}}d_{n+1}/at^{2^{n+1}} = (bf^{2^n}d_n/at^{2^n})^2([n+1]a/b)$$
>
> are integral. Hence the claim follows exactly as before. (Thakur, 1996, p. 255)

(The second claim of Theorem 3 is: If, in addition t divides b, (and does not divide a) then $N := (a/b)e(t/f) + c/d$ is also of pure e-type.)

$(\frac{bf^{2^n}d_n}{at^{2^n}})^2(\frac{[n+1]a}{b})$ is an integer because $t^{2^n}|[n+1]$ and $n-1$ was chosen to be a positive integer larger than the degree of a.

Observe this for $n = 4$.

$(bf^{2^n}d_n/at^{2^n})^2 = (bf^{2^4}d_4/at^{2^4})^2 = (\frac{(t^2+t)(t+1)^{16}(t^{16}+t)(t^8+t)^2(t^4+t)^4(t^2+t)^8}{(t+1)t^{16}})^2$

$= (\frac{(t^{16}+t)(t^8+t)^2(t^4+t)^4(t^2+t)^9(t+1)^{15}}{t^{16}})^2 = \{0, 16, 24, 28, 30, 32, 40, 44, 46, 48, 52, 54, 56, 58, 60, 62, 68, 70, 72, 74, 76, 78, 82, 84, 86, 90, 98, 100, 102, 106, 114, 130\}$, which is integral.

$([n+1]a/b) = ([5]a/b) = \frac{[5](t+1)}{t^2+t} = \frac{(t^{32}+t)(t+1)}{t^2+t} = t^{31}+1$, which is integral.

> Now note that for large enough n, the truncated series (displayed in the proof of the last theorem) for M give convergents (i.e. truncated continued fractions) for M because of rapid convergence. (More precisely, if the truncation is p/q with $(p,q) = 1$, then $\deg(M - p/q) < -2 \deg q$, by straightforward calculation, and (e.g. by Lemma 1 of [BS]) this guarantees that p/q is a convergent). If b is square-free, let m be the least common multiple of the degrees of the primes dividing b. Then by 3.1.1, when m divides $n+1$, then b divides $[n+1]$ and Lemma 1 gives the reversal as claimed at the corresponding truncations. □ (Thakur, 1996, p. 255)

Now, step through the statements of the last paragraph with the example (FHE2-1), Appendix E.7, p. 95.

First, look at various a_n, p_n and q_n for the example. It is seen that $q = 2$, $x = t^2$, $y = t^2 + 1$, $z = 0$, $w = 1$, $\theta = t + 1$, $f = t$.

$(\frac{x}{y})e(\frac{\theta}{f}) + (\frac{z}{w}) = (\frac{t^2}{t^2+1})e(\frac{t+1}{t})$

$a_0 = 1$

$a_1 = t = \{1\}$

$a_2 = t^3 + t^6 = \{3, 6\}$

$a_3 = t = \{1\}$

$a_4 = 1 + t + t^2 + t^4 + t^6 + t^8 = \{0, 1, 2, 4, 6, 8\}$

$a_5 = t + t^2 = \{1, 2\}$

$p_0 = a_0 = 1$

$p_1 = a_1 a_0 + 1 = 1 + t = \{0, 1\}$

$p_2 = a_2 p_1 + p_0 = 1 + t^3 + t^4 + t^6 + t^7 = \{0, 3, 4, 6, 7\}$

$p_3 = a_3 p_2 + p_1 = 1 + t^4 + t^5 + t^7 + t^8 = \{0, 4, 5, 7, 8\}$

$p_4 = a_4 p_3 + p_2 = t + t^2 + t^3 + t^4 + t^7 + t^9 + t^{14} + t^{15} + t^{16}$
$\quad = \{1, 2, 3, 4, 7, 19, 14, 15, 16\}$

$p_5 = a_5 p_4 + p_3 = 1 + t^2 + t^4 + t^5 + t^6 + t^7 + t^9 + t^{11} + t^{15} + t^{18} =$
$\quad = \{0, 2, 4, 5, 6, 7, 9, 11, 15, 18\}$

$q_0 = 1$

$q_1 = a_1 = t = \{1\}$

$q_2 = a_2 q_1 + q_0 = 1 + t^4 + t^7 = \{0, 4, 7\}$

$q_3 = t^5 + t^8 = \{5, 8\}$

$q_4 = 1 + t^4 + t^5 + t^6 + t^8 + t^{10} + t^{11} + t^{12} + t^{13} + t^{14} + t^{16}$

$\quad = \{0, 4, 5, 6, 8, 10, 11, 12, 13, 14, 16\}$

$q_5 = t + t^2 + t^9 + t^{10} + t^{11} + t^{16} + t^{17} + t^{18}$

$\quad = \{1, 2, 9, 10, 11, 16, 17, 18\}$

Approximate M as $(\frac{t^2}{t^2+1}) \sum_{i=1}^{8} \frac{(\frac{t+1}{t})^{q^i}}{D_i}$. Then,

$$M \approx \frac{1+t^{256}+t^{320}+\cdots+t^{2046}+t^{2047}+t^{2048}}{t^{509}+t^{511}+t^{637}+\cdots+t^{1920}+t^{2046}+t^{2048}}$$

Setting $n = 5$,

$$M - \frac{p_5}{q_5} \approx \frac{1+t^{256}+t^{320}+\cdots+t^{2046}+t^{2047}+t^{2048}}{t^{509}+t^{511}+t^{637}+\cdots+t^{1920}+t^{2046}+t^{2048}} - \frac{1+t^2+t^4+t^5+t^6+t^7+t^9+t^{10}+t^{11}+t^{15}+t^{18}}{t+t^2+t^9+t^{10}+t^{11}+t^{16}+t^{17}+t^{18}}$$

$$\approx \frac{1+t^8+t^{10}+\cdots+t^{2018}+t^{2020}+t^{2024}}{t^{509}+t^{511}+t^{517}\cdots t^{2057}+t^{2061}+t^{2064}}$$

Note that deg $q_5 = 18$. The deg $(M - \frac{p_5}{q_5}) \approx$ -40 $< -2 * 18$

So, with the stated example, for $n = 5$, deg(M-p/q) < -2 deg q.

Now, provide an example incorporating the following. Let b be square-free and m be the least common multiple of the degrees of the primes dividing b. Use the definition of $[i]$, the product of monic irreducible polynomials of degree dividing i, found in Chapter 2.1.2, p. 6. Note that when m divides $n + 1$, then b divides $[n + 1]$ and Lemma 1 gives the reversal as claimed at the corresponding truncations.

Let $b = t(t + 1)(t^2 + t + 1) = t^4 + t$.

Then, m = LCM(1, 1, 2) = 2. $m|(3 + 1)$, so choose n to be 3.

$$\frac{[n+1]}{t^4+t} = \frac{[4]}{t^4+t} = \frac{t^{16}-1}{t^4+t} = \frac{t^{16}+1}{t^4+t} = 1 + t^3 + t^6 + t^9 + t^{12}.$$

So, for this example, it has been seen that $b|[n + 1]$.

3.2.6.3 Th(4.6.) Example of Theorem 3.

Let $a = t + 1$, $b = t^2 + t$, $c = 1$, $d = t$ and $f = t + 1$.

Note that t divides b, t does not divide a, and b is a square-free polynomial.

According to the theorem, M and N will be of "e-type," and for infinitely many n, the CF for M is of the form $[a_0, \overrightarrow{X}, a_n, -\overleftarrow{X}, ...]$ with $\overrightarrow{X} = (a_1, ..., a_{n-1})$.

$M := (\frac{a}{b})e(\frac{1}{f}) + (\frac{c}{d}) = (\frac{t+1}{t^2+t})e(\frac{1}{t+1}) + (\frac{1}{t})$

The first 65 terms of M are:

[0, {0, 1}, {2, 3}, {0, 1}, {0, 3}, {0, 1}, {2, 3}, {0, 1}, {0, 7},
{0, 1}, {2, 3}, {0, 1}, {0, 3}, {0, 1}, {2, 3}, {0, 1}, {0, 15},
{0, 1}, {2, 3}, {0, 1}, {0, 3}, {0, 1}, {2, 3}, {0, 1}, {0, 7},
{0, 1}, {2, 3}, {0, 1}, {0, 3}, {0, 1}, {2, 3}, {0, 1}, {0, 31},
{0, 1}, {2, 3}, {0, 1}, {0, 3}, {0, 1}, {2, 3}, {0, 1}, {0, 7},
{0, 1}, {2, 3}, {0, 1}, {0, 3}, {0, 1}, {2, 3}, {0, 1}, {0, 15},
{0, 1}, {2, 3}, {0, 1}, {0, 3}, {0, 1}, {2, 3}, {0, 1}, {0, 7},
{0, 1}, {2, 3}, {0, 1}, {0, 3}, {0, 1}, {2, 3}, {0, 1}, {0, 63}, ...]

$N := (\frac{a}{b})e(\frac{1}{f}) + (\frac{c}{d}) = (\frac{t+1}{t^2+t})e(\frac{t}{t+1}) + (\frac{1}{t})$

The first 65 terms of N are:

[0, {2}, {0, 1}, {1}, {1}, {0, 1}, {2, 3}, {0, 1}, {5},
{1}, {0, 1}, {2}, {0, 3}, {2}, {0, 1}, {1}, {13},
{0, 1}, {2, 3}, {0, 1}, {1}, {1}, {0, 1}, {2}, {0, 7},
{2}, {0, 1}, {1}, {1}, {0, 1}, {2, 3}, {0, 1}, {29},
{1}, {0, 1}, {2}, {0, 3}, {2}, {0, 1}, {1}, {5},
{0, 1}, {2, 3}, {0, 1}, {1}, {1}, {0, 1}, {2}, {0, 15},
{2}, {0, 1}, {1}, {1}, {0, 1}, {2, 3}, {0, 1}, {5},
{1}, {0, 1}, {2}, {0, 3}, {2}, {0, 1}, {1}, {61}, ...]

3.2.7 Th(4.7.) More Fancy Numbers and Subtle Pattern Variations, $q = 2$

With this subsection, Dr. Thakur proves lemmas and theorems which provide fancy examples of numbers in cardinality $q = 2$ with very subtle pattern variations. These include e/t^n and $e/\overline{p}$, where $\overline{p}$ is the degree two prime.

3.2.7.1 Th(4.7.) Remark.

Here, theta (θ), in simple Hurwitz numbers of the form $(a/b)e(\theta/f) = c/d$, is addressed. When cardinality $q > 2$, theta is always equal to 1, and simple Hurwitz numbers have the form $(a/b)e(1/f) = c/d$. Part of what the following remark says is that when cardinality $q = 2$, theta can be replaced with "t," "$t + 1$" or "$t^2 + t$." An example, then, of a simple Hurwitz number in this case, would be $(a/b)e((t+1)/f) = c/d$.

Dr. Thakur again emphasizes that patterns for cardinality $q = 2$ are more subtle than those of $q > 2$, with reversing and repeating of patterns. Here, he also sets the stage for looking at exponentials with denominators with higher multiplicities and degrees.

> *Remark*. We can of course replace t by $t + 1$ or $t^2 + t$ in the theorem above. If we look for the analogue of $2/n$ in the Hurwitz theorem, at first we may think of α/f with $\alpha \in \mathbb{F}_q$ (in particular, say $q - 1$, which plays the role of 2, in many contexts). Indeed, since $e(\alpha/f) = \alpha e(1/f)$ such a result is included in our result. but it seems that the fact that 1 and 2 are the only allowed numerators in the classical case is related to the fact that only first and second roots of unity are in the ground field $\mathbb{Q}$. With this interpretation, if we look for the a-torsion points of the Carlitz module (see the introduction of [T1] or [GHR]; these are analogues of the roots of unity), which are in K, then an easy calculation shows that for $q > 2$ this forces $a \in \mathbb{F}_q$, but for $q = 2; a = t, t + 1$, or $t^2 + t$ are exactly the extra a's that are allowed. This fits with our result.
>
> Now we show that the patterns in the general case are more subtle when $q = 2$, giving a mixture of reversing and repeating of patterns, and we do not have the same results as for $q > 2$. Since we have some general results in the previous theorem about square-free denominators and complete result for a degree one denominator, we now look at denominators with higher multiplicity and degrees. (Thakur, 1996, p. 255)

3.2.7.2 Th(4.7.) Theorem 4.

With Theorem 4, Dr. Thakur studies an exponential with a denominator with higher multiplicity and degree. In this case, he looks at e/t^n.

> THEOREM 4. Let $q = 2$. Then for $n > 2$, and with $\overrightarrow{X_n}$ defined by $\sum_{i=0}^{n-2} 1/(d_i t^n) = [0, \overrightarrow{X_n}]$ we have,
>
> $$\tfrac{e}{t^n} = [0, \overrightarrow{X_n}, t^{2^{n-1}-n}, \overrightarrow{X_n}, t^{2^n-n}, \overrightarrow{X_n}, t^{2^{n-1}-n}, \overrightarrow{X_n}, t^{2^{n+1}-n}, ...]$$
>
> Also, with $\overrightarrow{X} = (t^2 + 1, t, t + 1)$ we have
>
> $$\tfrac{e}{t^2} = [0, \overrightarrow{X}, t^2, \overrightarrow{X}, t^6, \overrightarrow{X}, t^2, \overrightarrow{X}, t^{14}, ...].$$
>
> (Thakur, 1996, p. 255)

3.2.7.3 Th(4.7.) Proof of Theorem 4.

Dr. Thakur begins the proof of this theorem by proving the expansion for e/t^2. He does this with the introduction and proof of "Lemma 2." Then, he goes on to prove the general case for e/t^n, with $n > 2$. He does this with the assistance of a "Claim."

> *Proof*. We start with e/t^2: Observe that $[0, \overrightarrow{X}] = 1/t^2 + 1/(t^2 d_1)$ and $[0, \overrightarrow{X}] + [0, \overleftarrow{X}] = 1/t$. (Thakur, 1996, p. 256)

$[0, \overrightarrow{X}] = \frac{1}{t^2} + \frac{1}{t^2 d_1} = \frac{1}{t^2} + \frac{1}{t^2(t^2+t)} = \frac{1+t+t^2}{t^3+t^4} = [0, 1 + t^2, t, 1 + t]$
$[0, \overleftarrow{X}] = [0, 1 + t, t, 1 + t^2]$

3.2.7.4 Th(4.7.) Lemma 2.

Lemma 2 speaks to how one convergent (i.e., truncated continued fraction) progresses to the next step as n goes from 1 to ∞. A similar stepwise progression is seen with Lemma 1.

LEMMA 2. Let $q = 2$ and $\overrightarrow{X} = (a_1, ..., a_n)$, so that $[0, \overrightarrow{X}] = p_n/q_n$. Then

$$[0, \overrightarrow{X}, \text{y}, \overrightarrow{X}] = p_n/q_n + 1/(q_n^2(y + (p_n + q_{n-1})/q_n))$$
$$= [0, \overrightarrow{X}] = 1/q_n^2(y + [0, \overrightarrow{X}] + [0, \overleftarrow{X}]).$$

(Thakur, 1996, p. 256)

3.2.7.5 Th(4.7.) Proof of Lemma 2.

The proof of Lemma 2 is based on the standard facts of continued fractions. Dr. Thakur's proof is corroborated here step by step, and an example with the continued fraction expansion of e/t^2 is provided.

Proof. The proof is a straightforward application of 1.1.1 − 1.1.5. In more detail

$$p_{n+1}/q_{n+1} := [0, \overrightarrow{X}, y, \overrightarrow{X}] = [0, \overrightarrow{X}, y + p_n/q_n]$$

(Thakur, 1996, p. 256)

Since, $[0, \overrightarrow{X}] = p_n/q_n$,

$[\text{y}, \overrightarrow{X}] = y + p_n/q_n$, and $[0, \overrightarrow{X}, y, \overrightarrow{X}] = [0, \overrightarrow{X}, y + p_n/q_n]$.

Continuing, from continued fraction fact 1.1.1, Chapter 2.2.1, p. 10,

$\frac{p_{n+1}}{q_{n+1}} = \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}}$

Then, with manipulation,

$\frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}} = \frac{p_n}{q_n} + \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n}$

$= \frac{p_n}{q_n} + \frac{q_n(a_{n+1}p_n + p_{n-1}) - p_n(a_{n+1}q_n + q_{n-1})}{q_n(a_{n+1}q_n + q_{n-1})}$

$= \frac{p_n}{q_n} + \frac{q_n(a_{n+1}p_n + p_{n-1}) - p_n(a_{n+1}q_n + q_{n-1})}{q_n^2(a_{n+1} + \frac{q_{n-1}}{q_n})}$

Use continued fraction fact 1.1.3, Chapter 2.2.1, p. 10, to simplify the above numerator and show that:

$q_n(a_{n+1}p_n + p_{n-1}) - p_n(a_{n+1}q_n + q_{n-1}) = (-1)^n$.

Since $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$, $p_{n+1} q_n - p_n q_{n+1} = (-1)^n$.

Reordering the terms gives $q_n p_{n+1} - p_n q_{n+1} = (-1)^n$, which leads easily to $q_n(a_{n+1} p_n + p_{n-1}) - p_n(a_{n+1} q_n + q_{n-1}) = (-1)^n$

Therefore,

$p_{n+1}/q_{n+1} = p_n/q_n + (-1)^n/(q_n^2(a_{n+1} + q_{n-1}/q_n))$.

But, in the stated case, $a_{n+1} = y + p_n/q_n$. So,

$[0, \overrightarrow{X}, \text{y}, \overrightarrow{X}] = p_n/q_n + 1/(q_n^2(y + (p_n + q_{n-1})/q_n))$

This proves the first equality. The second equality follows from continued fraction fact 1.1.5, Chapter 2.2.1, p. 10.

Continuing, show that $\frac{p_n + q_{n-1}}{q_n} = [0, \overrightarrow{X}] + [0, \overleftarrow{X}]$. Since $\frac{q_n}{q_{n-1}} = \overleftarrow{X}$, $\frac{q_{n-1}}{q_n} = [0, \overleftarrow{X}]$. Note that $[0, \overleftarrow{X}]$ is the reciprocal of $\overleftarrow{X}$, with $[0, \overleftarrow{X}] < 1$ and $\overleftarrow{X} > 1$. Then, $\frac{p_n + q_{n-1}}{q_n} = [0, \overrightarrow{X}] + [0, \overleftarrow{X}]$.

Therefore,

$[0, \overrightarrow{X}, \text{y}, \overrightarrow{X}] = [0, \overrightarrow{X}] + 1/q_n^2(y + [0, \overrightarrow{X}] + [0, \overleftarrow{X}])$. □

Let us write $p_{k_n}/q_{k_n} = 1/t^2 + \cdots + 1/(t^2 d_n) = [0, \overrightarrow{Y_n}]$, it being understood as usual that p_{k_n} and q_{k_n} are relatively prime. Then $q_{k_n} = t^2 d_n$ is also the denominator of $q_{k_n-1}/q_{k_n} = [0, \overleftarrow{Y_n}]$ (by 1.1.5). Let our induction hypothesis be that $p_{k_n}/q_{k_n} + q_{k_n-1}/q_{k_n} = 1/t$. This is true for $n = 1$. Lemma 2 and 3.1.2 together imply that

$$[0, \overrightarrow{Y_n}, t^{2^{n+1}-2}, \overrightarrow{Y_n}] = p_{k_n}/q_{k_n} + 1/(t^2 d_{n+1}) = p_{k_{n+1}}/q_{k_{n+1}}$$

(notice that this is the claim in the statement of the Theorem, so we are really using double induction) and similarly also that

$$q_{k_{n+1}-1}/q_{k_{n+1}} = [0, \overleftarrow{Y_n}, t^{2^{n+1}-2}, \overleftarrow{Y_n}] = q_{k_n-1}/q_{k_n} + 1/t^2(d_{n+1})$$

(Thakur, 1996, p. 256)

Observe why

$[0, \overrightarrow{Y_n}, t^{2^{n+1}-2}, \overrightarrow{Y_n}] = p_{k_n}/q_{k_n} + 1/(t^2 d_{n+1}) = p_{k_{n+1}}/q_{k_{n+1}}$

From Lemma 2, it is seen that:

$[0,\ \overrightarrow{Y_n},\ t^{2^{n+1}-2},\ \overrightarrow{Y_n}] = p_{k_n}/q_{k_n} + \frac{1}{q_{k_n}^2(t^{2^{n+1}-2}+\frac{p_{k_n}+q_{k_n-1}}{q_{k_n}})}$

It is given that $q_{k_n} = t^2 d_n$ and that $p_{k_n}/q_{k_n} + q_{k_n-1}/q_{k_n} = 1/t$. So,

$p_{k_n}/q_{k_n} + \frac{1}{q_{k_n}^2(t^{2^{n+1}-2}+\frac{p_{k_n}+q_{k_n-1}}{q_{k_n}})} = p_{k_n}/q_{k_n} + \frac{1}{(t^2 d_n)^2(t^{2^{n+1}-2}+\frac{1}{t})}$

$= p_{k_n}/q_{k_n} + \frac{1}{t^4 d_n^2(t^{2^{n+1}-2}+\frac{1}{t})} = p_{k_n}/q_{k_n} + \frac{1}{t^2 d_n^2(t^{2^{n+1}}+t)} = p_{k_n}/q_{k_n} + \frac{1}{t^2 d_n^2[n+1]}$

Since $d_{n+1} = [n+1]d_n^2$,

$[0,\ \overrightarrow{Y_n},\ t^{2^{n+1}-2},\ \overrightarrow{Y_n}] = p_{k_n}/q_{k_n} + 1/(t^2 d_{n+1}) = p_{k_{n+1}}/q_{k_{n+1}}$ □

The fact that $q_{k_{n+1}-1}/q_{k_{n+1}} = [0,\ \overleftarrow{Y_n},\ t^{2^{n+1}-2},\ \overleftarrow{Y_n}] = q_{k_n-1}/q_{k_n} + 1/t^2(d_{n+1})$ follows directly from the expansion for $p_{k_{n+1}}/q_{k_{n+1}}$ and continued fraction fact 1.1.5, Chapter 2.2.1, p. 10.

The first 65 terms of the simple CF expansion of $\frac{e}{t^2}$ are the following:

$\frac{e}{t^2}$ = [0,
{0, 2}, {1}, {0, 1}, {2}, {0, 2}, {1}, {0, 1}, {6},
{0, 2}, {1}, {0, 1}, {2}, {0, 2}, {1}, {0, 1}, {14},
{0, 2}, {1}, {0, 1}, {2}, {0, 2}, {1}, {0, 1}, {6},
{0, 2}, {1}, {0, 1}, {2}, {0, 2}, {1}, {0, 1}, {30},
{0, 2}, {1}, {0, 1}, {2}, {0, 2}, {1}, {0, 1}, {6},
{0, 2}, {1}, {0, 1}, {2}, {0, 2}, {1}, {0, 1}, {14},
{0, 2}, {1}, {0, 1}, {2}, {0, 2}, {1}, {0, 1}, {6},
{0, 2}, {1}, {0, 1}, {2}, {0, 2}, {1}, {0, 1}, {62}, ...]

This shows that the induction hypothesis is true in general and consequently, the claim in the case of e/t^2 follows.

For the general case $n > 2$,

Claim. $[0,\ \overrightarrow{X_n}] + [0,\ \overleftarrow{X_n}] = 1/t^{n-1}$.

Assuming this, by induction using the lemma just as before we get the formula claimed for e/t^n, so it remains to prove the claim: We have $q_{k_n} = t^n d_{n-2}$ and $p_{k_n} = d_{n-2}(1 + 1/d_1 + \cdots + 1/d_{n-2})$. Let $Q = td_{n-2} + p_{k_n}$ and $P = (1 + p_{k_n}Q)/q_{k_n}$. Then $Pq_{k_n} - Qp_{k_n} = 1$. We first show that P is an integer. Now

$$P = \frac{1+p_{k_n}^2+td_{n-2}P_{k_n}}{t^n d_{n-2}}$$
$$= \frac{(d_{n-2}+\frac{d_{n-2}}{d_1^2}+\cdots+\frac{d_{n-2}}{d_{n-3}^2})+t(d_{n-2}+\frac{d_{n-2}}{d_1}+\cdots+\frac{d_{n-2}}{d_{n-2}})}{t^n}$$

By 3.1.1 and 3.1.2, (we calculate directly for $n = 3$ and check that P is an integer) it follows that for $n > 3$, $td_{n-2} \equiv 0 \pmod{t^n}$ and

$$\frac{d_{n-2}}{d_{n-i}^2} + t\frac{d_{n-2}}{d_{n-i+1}} = \frac{d_{n-2}}{d_{n-i+1}}([n-i+1]+t)$$
$$= \frac{d_{n-2}}{d_{n-i+1}} t^{2^{n-i+1}} \equiv 0 \pmod{t^n}$$

Hence, it is seen that P is an integer.

Now $Pq_{k_n} - Qp_{k_n} = 1$, whereas by 1.1.3, we have $p_{k_n-1}q_{k_n} - q_{k_n-1}p_{k_n} = 1$, so that $p_{k_n}(q_{k_n-1} - Q) = q_{k_n}(p_{k_n-1} - P)$. Now p_{k_n} and q_{k_n} are relatively prime by continued fraction fact 1.1.3, and it follows easily from the definitions that the degree of $p_{k_n-1} - P$ is less than that of p_{k_n}. Hence $P = p_{k_n-1}$ and the latter formula proves the claim. Hence the proof of the claim and the theorem is complete. $\square$ (Thakur, 1996, ps. 256, 257)

See Appendix F, p. 103 to see the above general case argument worked through for the examples of $n = 3$ and $n = 4$.

Next we deal with the CF for $e/\overline{p}$, where $\overline{p} = t^2 + t + 1$, the degree two prime. Let $\theta_i := \sum_{j=0}^{i} 1/(\overline{p}d_j)$. (Thakur, 1996, p. 257)

3.2.7.6 Th(4.7.) Theorem 5.

Theorem 5 is the second theorem related to an exponential with a denominator with higher multiplicity and degree. This is a fun number to study, with very interesting properties, as shall be seen. Note the similarities between this number, $e/(t^2 + t + 1)$, and $\alpha = e/(t^3 + t + 1)$, as seen in Appendix H, p. 109.

Theorem 5. The CF for $e/(t^2 + t + 1)$ is μ_∞, which is a limit of its truncations μ_i defined as follows:

$$\mu_3 := [0, [1], [2]\,\overline{p}, [1], [2][1] + 1, [2], [1], [2]]$$

For odd $k \geq 3$, if $\mu_k = [0, \overrightarrow{Y}]$, let $\mu_{k+1} := [0, \overrightarrow{Y}, [k+1]/\overline{p}, \overleftarrow{Y}]$. For even $k \geq 4$, if $\mu_k = [0, \overrightarrow{Z}, [k]/\overline{p}, \overleftarrow{Z}]$, let

$$\mu_{k+1} := [0, \overrightarrow{Z}, [k]/\overline{p}, \overleftarrow{Z}, \lfloor [k+1]/\overline{p} \rfloor, \overleftarrow{Z}, [k]/\overline{p}, \overrightarrow{Z}]$$

where $\lfloor [k+1]/\overline{p} \rfloor$ denotes the polynomial obtained as the quotient when $[k+1]$ is divided by $\overline{p}$ using the division algorithm. (Thakur, 1996, p. 257)

For further development of $e/(t^2+t+1)$, see Appendix G on p. 105.

3.2.7.7 Th(4.7.) Proof of Theorem 5.

Dr. Thakur proves this theorem by showing that $\mu_k = \theta_k$ for $k \geq 3$. He does this with the assistance of "Lemma 3."

Proof. First observe that μ_k is well defined. It is enough to prove that $\mu_k = \theta_k$ for $k \geq 3$. The proof is by induction on k and holds for $k = 3$ by construction. (Thakur, 1996, p. 257)

From Appendix G on p. 106, it is seen that

$$\begin{aligned}
\mu_3 &:= [0, [1], [2]\,\overline{p}, [1], [2][1]+1, [2], [1], [2]] \\
&= [0, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{0, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\}]
\end{aligned}$$

$$\begin{aligned}
\theta_3 &:= \tfrac{1}{(t^2+t+1)d_0} + \tfrac{1}{(t^2+t+1)d_1} + \tfrac{1}{(t^2+t+1)d_2} + \tfrac{1}{(t^2+t+1)d_3} \\
&= \tfrac{1}{t^2+t+1} + \tfrac{1}{(t^2+t+1)[1]} + \tfrac{1}{(t^2+t+1)[2][1]^2} + \tfrac{1}{(t^2+t+1)[3][2]^2[1]^4} \\
&= \tfrac{1+t^4+t^7+t^8+t^{12}+t^{17}+t^{18}+t^{19}+t^{21}+t^{22}+t^{24}}{t^7+t^8+t^9+t^{11}+t^{12}+t^{16}+t^{17}+t^{21}+t^{22}+t^{24}+t^{25}+t^{26}} \\
&= [0, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{0, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\}]
\end{aligned}$$

For odd k the passage from k to $k+1$ follows by Lemma 1. (Thakur, 1996, p. 257)

This is simply because, with $q = 2$, Lemma 1 states that:

Let $[0, \overrightarrow{Y}] = \frac{p_k}{q_k}$. Then, $[0, \overrightarrow{Y}, \frac{[k+1]}{\overline{p}}, \overleftarrow{Y}] = \frac{p_k}{q_k} + \frac{1}{\frac{[k+1]}{\overline{p}} q_k^2}$.

Well, $\frac{p_k}{q_k} + \frac{1}{\frac{[k+1]}{\overline{p}} q_k^2} = \sum_{j=0}^{k+1} \frac{1}{\overline{p} d_j} = \theta_{k+1}$.

Observe this for $k = 3$. From Appendix G on p. 106, it is seen that p_k correlates to p_{n_3} and q_k correlates to q_{n_3}. Additionally,

$p_{n_3} = 1 + t^4 + t^7 + t^8 + t^{12} + t^{17} + t^{18} + t^{19} + t^{21} + t^{22} + t^{24}$
$q_{n_3} = t^7 + t^8 + t^9 + t^{11} + t^{12} + t^{16} + t^{17} + t^{21} + t^{22} + t^{24} + t^{25} + t^{26}$

and

$\mu_3 = \frac{p_{n_3}}{q_{n_3}} = \frac{1+t^4+t^7+t^8+t^{12}+t^{17}+t^{18}+t^{19}+t^{21}+t^{22}+t^{24}}{t^7+t^8+t^9+t^{11}+t^{12}+t^{16}+t^{17}+t^{21}+t^{22}+t^{24}+t^{25}+t^{26}}$

$\frac{p_{n_3}}{q_{n_3}} + \frac{1}{\frac{[k+1]}{\overline{p}} q_{n_3}^2} = \frac{1+t^4+t^7+t^8+t^{12}+t^{17}+t^{18}+t^{19}+t^{21}+t^{22}+t^{24}}{t^7+t^8+t^9+t^{11}+t^{12}+t^{16}+t^{17}+t^{21}+t^{22}+t^{24}+t^{25}+t^{26}}$
$+ \frac{1}{\frac{[4]}{t^2+t+1}(t^7+t^8+t^9+t^{11}+t^{12}+t^{16}+t^{17}+t^{21}+t^{22}+t^{24}+t^{25}+t^{26})^2}$

= ({0, 8, 14, 19, 22, 23, 24, 27, 28, 29, 30, 31, 33, 34, 36, 37, 38, 40, 41, 44, 47, 48, 49, 51, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 64})/ ({15, 16, 17, 23, 24, 25, 27, 28, 32, 35, 36, 40, 41, 45, 46, 49, 53, 54, 56, 57, 58, 64, 65, 66}) (Note, this is a quotient.)

$= \frac{1}{(t^2+t+1)d_0} + \frac{1}{(t^2+t+1)d_1} + \frac{1}{(t^2+t+1)d_2} + \frac{1}{(t^2+t+1)d_3} + \frac{1}{(t^2+t+1)d_4}$
$= \frac{p_{n_3+1}}{q_{n_3+1}} = \mu_4$

> Now if we write $\mu_k = p_{n_k}/q_{n_k}$, then we simultaneously claim by induction that for odd k we have $(p_{n_k} + q_{n_k-1})/q_{n_k} = 1/\overline{p}$. Again $k = 3$ is a straightforward calculation. (Thakur, 1996, p. 258)

For an example of this, start with the evaluations of p_{n_3}, q_{n_3}, and μ_3, above.

From continued fraction fact 1.1.5, Chapter 2.2.1, p. 10, it is seen that

$\frac{q_{n_3}}{q_{n_3-1}} = \frac{t^7+t^8+t^9+t^{11}+t^{12}+t^{16}+t^{17}+t^{21}+t^{22}+t^{24}+t^{25}+t^{26}}{q_{n_3-1}} = [a_{n_3}, a_{n_3-1}, \dots, a_1]$

= [{1:1, 1:4}, {1:1, 1:2}, {1:1, 1:4}, {1, 1:2, 1:3, 1:5, 1:6}, {1:1, 1:2}, {1:1, 1:2, 1:3, 1:4, 1:5, 1:6}, {1:1, 1:2}]

$= \frac{t^7+t^8+t^9+t^{11}+t^{12}+t^{16}+t^{17}+t^{21}+t^{22}+t^{24}+t^{25}+t^{26}}{1+t^4+t^8+t^{11}+t^{12}+t^{13}+t^{14}+t^{19}+t^{20}+t^{21}+t^{22}}$

Solving for q_{n_3-1} gives

$q_{n_3-1} = 1 + t^4 + t^8 + t^{11} + t^{12} + t^{13} + t^{14} + t^{19} + t^{20} + t^{21} + t^{22}$

Then, it follows with computation that

$\frac{p_{n_3}+q_{n_3-1}}{q_{n_3}} =$

$\frac{(1+t^4+t^7+t^8+t^{12}+t^{17}+t^{18}+t^{19}+t^{21}+t^{22}+t^{24})+(1+t^4+t^8+t^{11}+t^{12}+t^{13}+t^{14}+t^{19}+t^{20}+t^{21}+t^{22})}{t^7+t^8+t^9+t^{11}+t^{12}+t^{16}+t^{17}+t^{21}+t^{22}+t^{24}+t^{25}+t^{26}}$

$= \frac{1}{1+t+t^2} = \frac{1}{p}$

which was to be shown for $k = 3$.

The induction on both claims is now complete by using the following lemma, in a similar manner as in the previous theorem. (The main calculation is spelled out after the proof of the Lemma). (Thakur, 1996, p. 258)

LEMMA 3. Let $q = 2$ and $\overrightarrow{X} = (a_1, ..., a_n)$, so that $[0, \overrightarrow{X}] = p_n/q_n$. Then

$$U := [0, \overrightarrow{X}, y, \overleftarrow{X}, x, \overleftarrow{X}, y, \overrightarrow{X}] = \frac{p_n}{q_n} + \frac{1}{yq_n^2} + \frac{1}{q_n^4 y^2 (x+(p_n+q_{n-1})/q_n)}$$

(Thakur, 1996, p. 258)

An example of this follows. Refer to the development of μ_3 with Appendix G, p. 106.

Let $q = 2$ and $\overrightarrow{X} = (a_1, ..., a_7)$

$= [\{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{0, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\}]$

so that $[0, \overrightarrow{X}]$

$= [0, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{0, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\}]$

$= \frac{p_7}{q_7} = \frac{1+t^4+t^7+t^8+t^{12}+t^{17}+t^{18}+t^{19}+t^{21}+t^{22}+t^{24}}{t^7+t^8+t^9+t^{11}+t^{12}+t^{16}+t^{17}+t^{21}+t^{22}+t^{24}+t^{25}+t^{26}}$

$= \frac{(\{0,4,7,8,12,17,18,19,21,22,24\})}{(\{7,8,9,11,12,16,17,21,22,24,25,26\})}$.

Let $x = (\{1, 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27, 29, 30\})$ and $y = (\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\})$. (Note, x and y are polynomials in t.)

Then, corroboration with Mathematica 5.2 shows that

$U := [0, \overrightarrow{X}, y, \overleftarrow{X}, x, \overleftarrow{X}, y, \overrightarrow{X}] = \mu_5$

$= [0, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{0, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\},$

$\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\},$

$\{1, 4\}, \{1, 2\}, \{1, 4\}, \{0, 2, 3, 5, 6\}, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\},$

$\{0, 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27, 29, 30\},$

$\{1, 4\}, \{1, 2\}, \{1, 4\}, \{0, 2, 3, 5, 6\}, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\},$

$\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\},$

{1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2}, {0, 2, 3, 5, 6}, {1, 4}]

= ({0, 16, 28, 31, 32, 35, 39, 40, 43, 44, 45, 46, 47, 48, 52, 53, 56, 57, 58, 59, 62, 63, 64, 65, 67, 68, 69, 71, 74, 75, 76, 78, 80, 81, 83, 85, 86, 87, 89, 93, 95, 97, 98, 99, 100, 101, 102, 104, 106, 107, 108, 110, 111, 116, 117, 119, 120, 122, 123, 125, 126, 129, 133, 135, 137, 138, 140, 143, 146, 151, 153, 154, 155, 156, 157, 158, 160})/ ({31, 32, 33, 47, 48, 49, 55, 56, 57, 59, 60, 64, 71, 72, 73, 75, 76, 80, 83, 84, 88, 89, 93, 94, 99, 100, 104, 105, 109, 110, 113, 117, 118, 120, 121, 122, 129, 133, 134, 136, 137, 138, 144, 145, 146, 160, 161, 162}) (Note, this is a quotient.)

$$= \frac{(\{0,4,7,8,12,17,18,19,21,22,24\})}{(\{7,8,9,11,12,16,17,21,22,24,25,26\})} + \frac{1}{(\{1,2,4,5,7,8,10,11,13,14\})(\{7,8,9,11,12,16,17,21,22,24,25,26\})^2} + [\frac{1}{(\{7,8,9,11,12,16,17,21,22,24,25,26\})^4(\{1,2,4,5,7,8,10,11,13,14\})^2} * \frac{1}{(\{0,2,3,5,6,8,9,11,12,14,15,17,18,20,21,23,24,26,27,29,30\}+\frac{p_7+q_6}{q_7})}]$$

$$= \frac{p_7}{q_7} + \frac{1}{yq_7^2} + \frac{1}{q_7^4 y^2 (x + \frac{p_7+q_6}{q_7})}$$

Note that $\frac{p_7+q_6}{q_7} = \frac{(\{0,4,7,8,12,17,18,19,21,22,24\})+(\{0,4,8,11,12,13,14,19,20,21,22\})}{(\{7,8,9,11,12,16,17,21,22,24,25,26\})} = \frac{1}{t^2+t+1}$

So, Lemma 3 is demonstrated for the example of μ_3 and μ_5 related to $\frac{e}{t^2+t+1}$ with $q = 2$, computed with Mathematica 5.2.

Proof. By Lemma 1 and 1.1.5, Chapter 2.2.1, p. 10, we have

$$\frac{p_m}{q_m} := [0, \overrightarrow{X}, y, \overleftarrow{X}] = \frac{p_n}{q_n} + \frac{1}{yq_n^2}, \qquad [0, \overleftarrow{X}, y, \overrightarrow{X}] = \frac{q_{n-1}}{q_n} + \frac{1}{yq_n^2}$$

Note that $q_m = yq_n^2$ and $p_m = 1 + yq_np_n$. Also by 1.1.5, we have $q_{m-1} = p_m$ by the reversal symmetry of the CF. Hence we have, by a calculation as in Lemma 2,

$$U = [0, \overrightarrow{X}, y, \overleftarrow{X}, x + \frac{q_{n-1}}{q_n} + \frac{1}{yq_n^2}]$$
$$= (\frac{p_n}{q_n} + \frac{1}{yq_n^2}) + \frac{1}{q_m^2(x + \frac{q_{n-1}}{q_n} + \frac{1}{yq_n^2} + \frac{q_{m-1}}{q_m})}$$

Substituting the values for q_m and q_{m-1} obtained above, it is seen that U is as claimed. This finishes the proof of the lemma. (Thakur, 1996, p. 258)

Observe this worked out for the example above. Note that $n = 7$ and $m = 15$.

$[0, \overrightarrow{X}] = [0, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{0, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\}]$

$= \frac{p_7}{q_7} = \frac{(\{0,4,7,8,12,17,18,19,21,22,24\})}{(\{7,8,9,11,12,16,17,21,22,24,25,26\})}.$

$y = (\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\}).$

$\frac{p_{15}}{q_{15}} := [0, \overrightarrow{X}, y, \overleftarrow{X}] = [0, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{1, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\},$

$\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\},$

$\{1, 4\}, \{1, 2\}, \{1, 4\}, \{1, 2, 3, 5, 6\}, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}]$

$= (\{0, 8, 14, 19, 22, 23, 24, 27, 28, 29, 30, 31, 33, 34, 36, 37, 38, 40, 41, 44, 47, 48, 49, 51, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 64\}) / (\{15, 16, 17, 23, 24, 25, 27, 28, 32, 35, 36, 40, 41, 45, 46, 49, 53, 54, 56, 57, 58, 64, 65, 66\})$ (Note, this is a quotient.)

$= \frac{(\{0,4,7,8,12,17,18,19,21,22,24\})}{(\{7,8,9,11,12,16,17,21,22,24,25,26\})}$

$+ \frac{1}{(\{1,2,4,5,7,8,10,11,13,14\})(\{7,8,9,11,12,16,17,21,22,24,25,26\})^2}$

$= \frac{p_7}{q_7} + \frac{1}{yq_7^2}$

$[0, \overleftarrow{X}, y, \overrightarrow{X}] = [0, \{1, 4\}, \{1, 2\}, \{1, 4\}, \{1, 2, 3, 5, 6\}, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\}, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{1, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\}]$

$= (\{0, 8, 14, 15, 19, 22, 24, 28, 31, 33, 34, 35, 36, 40, 42, 47, 48, 50, 51, 52, 53, 54, 55, 57, 58, 59, 60, 61, 62\}) / (\{15, 16, 17, 23, 24, 25, 27, 28, 32, 35, 36, 40, 41, 45, 46, 49, 53, 54, 56, 57, 58, 64, 65, 66\})$ (Note, this is a quotient.)

$= \frac{(\{0,4,8,11,12,13,14,19,20,21,22\})}{(\{7,8,9,11,12,16,17,21,22,24,25,26\})}$

$+ \frac{1}{(\{1,2,4,5,7,8,10,11,13,14\})(\{7,8,9,11,12,16,17,21,22,24,25,26\})^2}$

$= \frac{q_6}{q_7} + \frac{1}{yq_7^2}$

The above statements were corroborated with Mathematica 5.2.

Now we spell out the details of the application of the lemma: First let us write Q for the quantity $\lfloor [k+1]/\overline{p} \rfloor$. By an easy induction we see that $t^{2^{2n+1}}$ is congruent to $t+1$ modulo $\overline{p}$, so that $\overline{p}Q = [k+1]+1$, for k even. Next note that $\theta_{k+1} = \theta_{k-1} + \frac{1}{(\overline{p}d_k)} + \frac{1}{(\overline{p}d_{k+1})}$. In the application of the lemma, p_n/q_n is θ_{k-1}, so that $q_n = \overline{p}d_{k-1}$ and $\frac{1}{yq_n^2} = \frac{1}{(\overline{p}d_k)}$ (by 3.1.2) and

$$\frac{1}{y^2 q_n^4 (x + \frac{(p_n + q_{n-1})}{q_n})} = \frac{1}{\overline{p}d_k^2(\overline{p}Q+1)} = \frac{1}{\overline{p}d_k^2[k+1]} = \frac{1}{\overline{p}d_{k+1}}$$

Hence the induction and the proof of the theorem is complete. □ (Thakur, 1996, p. 258)

Now, corroborate the details of the application of the lemma accordingly.

First, show by induction that $t^{2^{2n+1}}$ is congruent to $t+1$ modulo $\overline{p}$. Dr. Thakur assisted me by e-mail with this analysis January 11, 2007.

First, let $n = 1$.

$t^{2^{2(n)+1}} = t^{2^{2(1)+1}} = t^{2^{2+1}} = t^8$

Now, $\overline{p}$ divides $t^4 + t$.

$\frac{t^4+t}{t^2+t+1} = t^2 + t$. So, $t^4 + t \cong 0$, modulo $\overline{p}$. Hence, $t^4 \cong t$, modulo $\overline{p}$. Also, $t^2 + t + 1 \cong 0$, modulo $\overline{p}$. Hence, $t^2 \cong t + 1$.

$t^8 = (t^4)^2 \cong t^2 \cong t + 1$, which was to be shown.

Secondly, assume that $t^{2^{2k+1}}$ is congruent to $t+1$ modulo $\overline{p}$. Show that $t^{2^{2(k+1)+1}}$ is congruent to $t + 1$ modulo $\overline{p}$.

$t^{2^{2k+1}} \cong t + 1$

$t + 1 \cong t^4 + 1 = (t + 1)^4 \cong (t^{2^{2k+1}})^4 = t^{4(2^{2k+1})} = t^{2^{2k+3}} = t^{2^{2(k+1)+1}}$, which was to be shown.

Let k be even. Then, there exists an integer m such that $2m = k$.

$[k + 1] = [2m + 1] = t^{2^{2m+1}} + t \cong t + 1 + t = 1$.

$[k + 1] + 1 \cong 0$.

$Q = \lfloor \frac{[k+1]}{\overline{p}} \rfloor = \lfloor \frac{1}{\overline{p}} \rfloor = 0$.

$\overline{p}Q = 0 \cong [k + 1] + 1$, modulo $\overline{p}$, for k even.

Continuing with the lemma application corroboration,

$\theta_{k+1} = \theta_{k-1} + \frac{1}{(\overline{p}d_k)} + \frac{1}{(\overline{p}d_{k+1})}$, by definition of the θ_i series, Chapter 3.2.7.5, p. 51.

The facts that $\frac{p_n}{q_n} = \theta_{k-1}$, $q_n = \overline{p}d_{k-1}$ and $\frac{1}{yq_n^2} = \frac{1}{(\overline{p}d_k)}$ are readily apparent.

So, $(yq_n^2)^2 = (\overline{p}d_k)^2 = \overline{p}^2 d_k^2$

Observe why $\overline{p}(x + \frac{(p_n+q_{n-1})}{q_n}) = \overline{p}Q + 1$. First, note that $x = Q = \lfloor \frac{[k+1]}{\overline{p}} \rfloor$. Also, it has been shown that $\frac{(p_n+q_{n-1})}{q_n} = \frac{1}{\overline{p}}$. So, $\overline{p}(x + \frac{(p_n+q_{n-1})}{q_n}) = \overline{p}(Q + \frac{1}{\overline{p}}) = \overline{p}Q + 1$

$= [k + 1]$ for k even.

So, the last line of the application of the lemma is readily apparent.

$$\frac{1}{y^2 q_n^4 (x + \frac{(p_n + q_{n-1})}{q_n})} = \frac{1}{\overline{p} d_k^2 (\overline{p} Q + 1)} = \frac{1}{\overline{p} d_k^2 [k+1]} = \frac{1}{\overline{p} d_{k+1}}$$

> These examples illustrate the subtleties of the case $q = 2$. We have some more examples of this kind and in all these examples, we do find some inductive scheme of block reversal and block repetition. But we have not yet understood the situation fully for general Hurwitz type numbers when $q = 2$. We hope to address that in a future paper. (Thakur, 1996, p. 259)

3.3 Special Phenomena

Dr. Thakur continues in this section by providing several examples of special phenomena. These include sequences which have pure "e-type" patterns.

> There are some special phenomena in characteristic p, which do not seem to have analogues for real numbers. (Thakur, 1996, p. 259)

3.3.1 Th(5.1.) $x^p = [(x_i^p)]$

With this subsection, Dr. Thakur states that in function fields with characteristic p, if you raise each partial quotient of a continued fraction expansion of the number x to the power p, you will have effectively raised the number x to the power p. Two examples of this have been provided.

> For example, $x = [(x_i)]$ implies $x^p = [(x_i^p)]$ in characteristic p, so nice patterns carry over for x^p from x. (Thakur, 1996, p. 259)

Observe two examples of this, first for characteristic $p = 2$, and secondly for characteristic $p = 3$.

First, look at the square of the degree two prime, which is examined in detail in Appendix G, p. 105.

The first 60 terms of its CF expansion are as follows:

$(\frac{e}{t^2+t+1})^2 = \frac{e^2}{t^4+t^2+1} =$

[0, {2, 4}, {2, 4, 6, 8, 10, 12}, {2, 4}, {0, 4, 6, 10, 12}, {2, 8}, {2, 4}, {2, 8},
{2, 4, 8, 10, 14, 16, 20, 22, 26, 28},
{2, 8}, {2, 4}, {2, 8}, {0, 4, 6, 10, 12}, {2, 4}, {2, 4, 6, 8, 10, 12}, {2, 4},
{0, 4, 6, 10, 12, 16, 18, 22, 24, 28, 30, 34, 36, 40, 42, 46, 48, 52, 54, 58, 60},
{2, 8}, {2, 4}, {2, 8}, {0, 4, 6, 10, 12}, {2, 4}, {2, 4, 6, 8, 10, 12}, {2, 4},
{2, 4, 8, 10, 14, 16, 20, 22, 26, 28},
{2, 4}, {2, 4, 6, 8, 10, 12}, {2, 4}, {0, 4, 6, 10, 12}, {2, 8}, {2, 4}, {2, 8}, ...] =

$[0^2, \{1, 2\}^2, \{1, 2, 3, 4, 5, 6\}^2, \{1, 2\}^2, \{0, 2, 3, 5, 6\}^2, \{1, 4\}^2, \{1, 2\}^2, \{1, 4\}^2,$
$\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\}^2,$
$\{1, 4\}^2, \{1, 2\}^2, \{1, 4\}^2, \{0, 2, 3, 5, 6\}^2, \{1, 2\}^2, \{1, 2, 3, 4, 5, 6\}^2, \{1, 2\}^2,$
$\{0, 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27, 29, 30\}^2,$
$\{1, 4\}^2, \{1, 2\}^2, \{1, 4\}^2, \{0, 2, 3, 5, 6\}^2, \{1, 2\}^2, \{1, 2, 3, 4, 5, 6\}^2, \{1, 2\}^2,$
$\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\}^2,$
$\{1, 2\}^2, \{1, 2, 3, 4, 5, 6\}^2, \{1, 2\}^2, \{0, 2, 3, 5, 6\}^2, \{1, 4\}^2, \{1, 2\}^2, \{1, 4\}^2, ...]$

Secondly, look at the cube of (FHE3-1), as seen in Appendix E.5, p. 92.

The first 31 terms of its CF expansion are as follows:

$((\frac{t+2}{2t})e(\frac{1}{t}) + \frac{t}{t+1})^3 = (\frac{t+2}{2t})^3(e(\frac{1}{t}))^3 + (\frac{t}{t+1})^3 = (\frac{t^3+2}{2t^3})(e(\frac{1}{t}))^3 + (\frac{t^3}{t^3+1}) =$

[{1:0}, {1:0, 1:3}, {1:0, 2:3}, {2:0, 1:3}, {1:0, 2:3}, {1:0, 2:3},
{2:12, 2:15, 2:18, 2:21, 2:24, 2:27, 1:36, 1:39, 1:42, 1:45, 1:48, 1:51},
{2:0, 1:3}, {2:0, 1:3}, {1:0, 2:3}, {2:0, 1:3}, {2:0, 2:3},
{1:39, 2:42, 2:57, 1:60, 2:63, 1:66, 1:81, 2:84, 2:117, 1:120, 1:135, 2:138, 1:141, 2:144, 2:159, 1:162},
{1:0, 1:3}, {1:0, 2:3}, {2:0, 1:3}, {1:0, 2:3}, {1:0, 2:3},
{1:12, 1:15, 1:18, 1:21, 1:24, 1:27, 2:36, 2:39, 2:42, 2:45, 2:48, 2:51},
{2:0, 1:3}, {2:0, 1:3}, {1:0, 2:3}, {2:0, 1:3}, {2:0, 2:3},
{2:120, 1:123, 1:174, 2:177, 1:192, 2:195, 1:198, 2:201, 2:246, 1:249, 2:252, 1:255, 2:270, 1:273, 1:324, 2:327, 1:360, 2:363, 2:414, 1:417, 2:432, 1:435, 2:438, 1:441, 1:486, 2:489, 1:492, 2:495, 1:510, 2:513, 2:564, 1:567},

{1:0, 1:3}, {1:0, 2:3}, {2:0, 1:3}, {1:0, 2:3}, {1:0, 2:3},

{2:12, 2:15, 2:18, 2:21, 2:24, 2:27, 1:36, 1:39, 1:42, 1:45, 1:48, 1:51}, ...] =

[$\{1{:}0\}^3$, $\{1{:}0,\ 1{:}1\}^3$, $\{1{:}0,\ 2{:}1\}^3$, $\{2{:}0,\ 1{:}1\}^3$, $\{1{:}0,\ 2{:}1\}^3$, $\{1{:}0,\ 2{:}1\}^3$,

$\{2{:}4,\ 2{:}5,\ 2{:}6,\ 2{:}7,\ 2{:}8,\ 2{:}9,\ 1{:}12,\ 1{:}13,\ 1{:}14,\ 1{:}15,\ 1{:}16,\ 1{:}17\}^3$,

$\{2{:}0,\ 1{:}1\}^3$, $\{2{:}0,\ 1{:}1\}^3$, $\{1{:}0,\ 2{:}1\}^3$, $\{2{:}0,\ 1{:}1\}^3$, $\{2{:}0,\ 2{:}1\}^3$,

$\{1{:}13,\ 2{:}14,\ 2{:}19,\ 1{:}20,\ 2{:}21,\ 1{:}22,\ 1{:}27,\ 2{:}28,\ 2{:}39,\ 1{:}40,\ 1{:}45,\ 2{:}46,\ 1{:}47,\ 2{:}48,\ 2{:}53,\ 1{:}54\}^3$,

$\{1{:}0,\ 1{:}1\}^3$, $\{1{:}0,\ 2{:}1\}^3$, $\{2{:}0,\ 1{:}1\}^3$, $\{1{:}0,\ 2{:}1\}^3$, $\{1{:}0,\ 2{:}1\}^3$,

$\{1{:}4,\ 1{:}5,\ 1{:}6,\ 1{:}7,\ 1{:}8,\ 1{:}9,\ 2{:}12,\ 2{:}13,\ 2{:}14,\ 2{:}15,\ 2{:}16,\ 2{:}17\}^3$,

$\{2{:}0,\ 1{:}1\}^3$, $\{2{:}0,\ 1{:}1\}^3$, $\{1{:}0,\ 2{:}1\}^3$, $\{2{:}0,\ 1{:}1\}^3$, $\{2{:}0,\ 2{:}1\}^3$,

$\{2{:}40,\ 1{:}41,\ 1{:}58,\ 2{:}59,\ 1{:}64,\ 2{:}65,\ 1{:}66,\ 2{:}67,\ 2{:}82,\ 1{:}83,\ 2{:}84,\ 1{:}85,\ 2{:}90,\ 1{:}91,\ 1{:}108,\ 2{:}109,\ 1{:}120,\ 2{:}121,\ 2{:}138,\ 1{:}139,\ 2{:}144,\ 1{:}145,\ 2{:}146,\ 1{:}147,\ 1{:}162,\ 2{:}163,\ 1{:}164,\ 2{:}165,\ 1{:}170,\ 2{:}171,\ 2{:}188,\ 1{:}189\}^3$,

$\{1{:}0,\ 1{:}1\}^3$, $\{1{:}0,\ 2{:}1\}^3$, $\{2{:}0,\ 1{:}1\}^3$, $\{1{:}0,\ 2{:}1\}^3$, $\{1{:}0,\ 2{:}1\}^3$,

$\{2{:}4,\ 2{:}5,\ 2{:}6,\ 2{:}7,\ 2{:}8,\ 2{:}9,\ 1{:}12,\ 1{:}13,\ 1{:}14,\ 1{:}15,\ 1{:}16,\ 1{:}17\}^3$, ...]

3.3.2 Th(5.2.) $e(t^{1/2}) - t^{1/2}$

With this subsection, Dr. Thakur shows that if you replace z in the exponential $e(z)$ with $t^{1/2}$ for $q = 2$, "$e(t^{1/2}) - t^{1/2}$" will have a pure e-type pattern. An example of this has been provided.

> For example, when $q = 2$, $e(t^{1/2}) - t^{1/2}$ has a pure e-type pattern as seen in section 4 as can be seen by putting $z = t^{1/2}$ in the Theorem 1. (Thakur, 1996, p. 259)

Using "Curly Bracket" notation defined in Appendix C.1 on p. 73, observe the following:

Let $q = 2$, and $z = t^{1/2}$. Then, $x_1 := [0,\ t^{(1/2)^{-2}}[1]] = [0,\ 1 + \frac{[1]}{t}] = [0,\ 1 + t]$.
$x_2 := [0,\ \frac{[1]}{T},\ [2],\ \frac{[1]}{T}] = [0,\ \{0,\ 1\},\ \{1,\ 4\},\ \{0,\ 1\}]$.
$e(t^{1/2}) - t^{1/2} = [0,$
{0, 1}, {1, 4}, {0, 1}, {1, 8}, {0, 1}, {1, 4}, {0, 1}, {1, 16},

{0, 1}, {1, 4}, {0, 1}, {1, 8}, {0, 1}, {1, 4}, {0, 1}, {1, 32},
{0, 1}, {1, 4}, {0, 1}, {1, 8}, {0, 1}, {1, 4}, {0, 1}, {1, 16},
{0, 1}, {1, 4}, {0, 1}, {1, 8}, {0, 1}, {1, 4}, {0, 1}, {1, 64},
{0, 1}, {1, 4}, {0, 1}, {1, 8}, {0, 1}, {1, 4}, {0, 1}, {1, 16},
{0, 1}, {1, 4}, {0, 1}, {1, 8}, {0, 1}, {1, 4}, {0, 1}, {1, 32},
{0, 1}, {1, 4}, {0, 1}, {1, 8}, {0, 1}, {1, 4}, {0, 1}, {1, 16},
{0, 1}, {1, 4}, {0, 1}, {1, 8}, {0, 1}, {1, 4}, {0, 1}, {1, 128}, ...]

Then, Dr. Thakur expands the statement to include various z and general q. An example with $z = t^{1/4}$ for $q = 2$ is provided.

> Similarly, if we subtract from $e(z)$ a few initial terms in the defining series we can get e-type patterns for resulting numbers for various z involving pth roots, for $q = 2$ or even for general q. We leave it to the reader to find such variants. (Thakur, 1996, p. 259)

An example of this follows. Evaluate $e(t^{1/4}) - t^{1/4}$ when $q = 2$:

The first few terms of $e(t^{1/4})$ look like the following:

$$e(t^{1/4}) = \frac{(t^{1/4})}{1} + \frac{(t^{1/4})^q}{D_1} + \frac{(t^{1/4})^{q^2}}{D_2} + \frac{(t^{1/4})^{q^3}}{D_3} + \frac{(t^{1/4})^{q^4}}{D_4} + \cdots$$
$$= \frac{(t^{1/4})}{1} + \frac{(t^{1/4})^2}{[1]} + \frac{(t^{1/4})^{2^2}}{[2][1]^2} + \frac{(t^{1/4})^{2^3}}{[3][2]^2[1]^{2^2}} + \frac{(t^{1/4})^{2^4}}{[4][3]^2[2]^{2^2}[1]^{2^3}} + \cdots.$$

Now, drop what were initially the x_0 and x_1 partial sums. It is seen that:

$$e(t^{1/4}) - t^{1/4} - t^{1/2} = \frac{(t^{1/4})^{2^2}}{[2][1]^2} + \frac{(t^{1/4})^{2^3}}{[3][2]^2[1]^{2^2}} + \frac{(t^{1/4})^{2^4}}{[4][3]^2[2]^{2^2}[1]^{2^3}} + \cdots$$

The first 65 terms of the simple CF expansion:

$e(t^{1/4}) - t^{1/4} - t^{1/2} = [0,$
{2, 4, 5, 7}, {1, 8}, {2, 4, 5, 7}, {1, 16},
{2, 4, 5, 7}, {1, 8}, {2, 4, 5, 7}, {1, 32},
{2, 4, 5, 7}, {1, 8}, {2, 4, 5, 7}, {1, 16},
{2, 4, 5, 7}, {1, 8}, {2, 4, 5, 7}, {1, 64},
{2, 4, 5, 7}, {1, 8}, {2, 4, 5, 7}, {1, 16},
{2, 4, 5, 7}, {1, 8}, {2, 4, 5, 7}, {1, 32},
{2, 4, 5, 7}, {1, 8}, {2, 4, 5, 7}, {1, 16},

$\{2, 4, 5, 7\}$, $\{1, 8\}$, $\{2, 4, 5, 7\}$, $\{1, 128\}$,
$\{2, 4, 5, 7\}$, $\{1, 8\}$, $\{2, 4, 5, 7\}$, $\{1, 16\}$,
$\{2, 4, 5, 7\}$, $\{1, 8\}$, $\{2, 4, 5, 7\}$, $\{1, 32\}$,
$\{2, 4, 5, 7\}$, $\{1, 8\}$, $\{2, 4, 5, 7\}$, $\{1, 16\}$,
$\{2, 4, 5, 7\}$, $\{1, 8\}$, $\{2, 4, 5, 7\}$, $\{1, 64\}$,
$\{2, 4, 5, 7\}$, $\{1, 8\}$, $\{2, 4, 5, 7\}$, $\{1, 16\}$,
$\{2, 4, 5, 7\}$, $\{1, 8\}$, $\{2, 4, 5, 7\}$, $\{1, 32\}$,
$\{2, 4, 5, 7\}$, $\{1, 8\}$, $\{2, 4, 5, 7\}$, $\{1, 16\}$,
$\{2, 4, 5, 7\}$, $\{1, 8\}$, $\{2, 4, 5, 7\}$, $\{1, 256\}$, ...]

REFERENCES

Baum, L., & Sweet, M. (1976). Continued fractions of algebraic power series in characteristic 2. *Annuls of Mathematics*, *103*, 593-610.

Dekking, M., Mendes, F. M., & Poorten, A. van der. (1982). Folds. *Mathematical Intelligencer*, *4*, 130-138, 173-181 and 190-195.

Dence, J. B., & Dence, T. P. (1999). *Elements of the theory of numbers*. Harcourt Academic Press.

Gallian, J. A. (2002). *Contemporary abstract algebra* (Fifth Edition ed.). Houghton Milton Company.

Goss, D. (1996). *Basic structures of function field arithmetic*. Springer-Verlag Berlin Heidelberg.

Goss, D., Hayes, D., & Rosen, M. (1992). *The arithmetic of function fields*. Berlin/New York: Gruyter.

Hardy, G. H., & Wright, E. M. (2000). *An introduction to the theory of numbers* (Fifth Edition ed.). Oxford University Press.

Poorten, A. van der, & Shallit, J. (1992). Folded continued fractions. *Journal of Number Theory*, *40*, 237-250. (Reprinted with permission from Elsevier.)

Shallit, J. (1979). Simple continued fractions for some irrational numbers, i. *Journal of Number Theory*, *11*, 209-217. (Reprinted with permission from Elsevier.)

Soanes, C., & Stevenson, A. (2004). *Concise oxford english dictionary* (11th ed.). Oxford University Press.

Thakur, D. S. (1992). Continued fraction for the exponential for $\mathbb{F}_q$[t]. *Journal of Number Theory*, *41*, 150-155. (Reprinted with permission from Elsevier.)

Thakur, D. S. (1996). Exponential and continued fractions. *Journal of Number Theory*, *59*(0097), 248-261. (Reprinted with permission from Elsevier.)

Thakur, D. S. (1997). Patterns of continued fractions for the analogues of e and related numbers in the function field case. *Journal of Number Theory*, *66*(NT972134), 129-147. (Reprinted with permission from Elsevier.)

Thakur, D. S. (2004). *Function field arithmetic*. World Scientific Publishing Co. Pte. Ltd.

Weisstein, E. W. (1999). *Crc concise encyclopedia of mathematics*. Chapman and Hall/CRC.

Zwillinger, D. (1996). *Crc standard mathematical tables and formulae* (30th ed.). CRC Press LLC.

Appendices

Appendix A

FUNCTION FIELD CONCEPTS/DEFINITIONS

This appendix highlights the development of two key function field concepts. The first is the field of p-adic numbers, $\mathbb{Q}_p$, and the second is the field of formal Laurent series over $\mathbb{F}_r$, $\mathbb{F}_r((1/T))$.

A.1 Development of the Field of p-Adic Numbers $\mathbb{Q}_p$

In his book, *Basic Structures of Function Field Arithmetic*, David Goss develops the concept of the field of p-adic numbers $\mathbb{Q}_p$. This field is the completion of the rational number fields.

> $K = \mathbb{Q}_p$, p a rational prime. Here one begins with the rational numbers $\mathbb{Q}$. If $x \in \mathbb{Q}^*$, then one decomposes x as
>
> $$x = p^t \cdot x_0,$$
>
> where $t \in \mathbb{Z}$ and neither the numerator nor denominator of x_0 involve p. One sets
>
> $$v(x) := v_p(x) := t.$$
>
> It is easy to see that v is a valuation. The field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to this valuation, just as $\mathbb{R}$ is obtained from $\mathbb{Q}$. From the third property of valuations, and for any $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, one has an equality of the sets
>
> $$|\mathbb{Q}_p|_v = |\mathbb{Q}|_v.$$
>
> This is clearly not true about the classical absolute value on $\mathbb{Q}$ and $\mathbb{R}$. (Goss, 1996, p. 36)

A.2 Development of $\mathbb{F}_r((1/T))$

David Goss goes on to develop the concept of $K = \mathbb{F}_r((1/T))$. This is the completion of the rational function fields, and is an analog of $\mathbb{R}$ (or $\mathbb{Q}_p$).

> $K = \mathbb{F}_r((\frac{1}{T}))$ = the field of formal Laurent series over $\mathbb{F}_r$ = finite field with r elements. The process of constructing K mirrors that of Part 1. One begins with $k = \mathbb{F}_r(T)$. One sets $v(0) = \infty$. If $x \in k^*$ is written
>
> $$(\tfrac{1}{T})^e \, x_0,$$
>
> with the numerator and denominator of x_0 prime to T, then one sets
>
> $$v(x) = e.$$
>
> The reader may then easily check that K is the completion of k with respect to $v(x)$. (N.B., one normally works with the isomorphic (under $T \leftrightarrow 1/T$) field $\mathbb{F}_r((T))$. For our purposes the formulation $\mathbb{F}_r((\frac{1}{T}))$ is best.) The valuation v extends continuously to K, etc. (Goss, 1996, p. 36)

Appendix B

ABSTRACT ALGEBRA DEFINITIONS

This appendix contains basic abstract algebra definitions. It is included as a background to the more complex function field topics discussed in this thesis.

B.1 Algebraic (Algebraic Extension)

Let E be an extension field of a field F and let $a \in E$. We call *a algebraic over F* if a is the zero of some nonzero polynomial in $F[x]$. If a is not algebraic over F, it is called *transcendental over F*. An extension E of F is called an *algebraic extension* of F if every element of E is algebraic over F. If E is not an algebraic extension of F, it is called a *transcendental* extension of F. An extension of F of the form $F(a)$ is called a *simple* extension of F. (Gallian, 2002, p. 361)

B.2 Algebraically Closed

A field that has no proper algebraic extension is called *algebraically closed*. (Gallian, 2002, p. 369)

Example: The set of complex numbers is the completion of an algebraic closure of the real numbers.

B.3 Cardinality

Two definitions related to “cardinality” are listed here.

cardinality: n. (pl. cardinalities) *Mathematics* the number of elements in a particular set or other grouping. (Soanes & Stevenson, 2004)

Cardinal Number

> In informal usage, a cardinal number is a number used in counting (a COUNTING NUMBER), such as 1, 2, 3, ... (Weisstein, 1999, p. 189)

Dr. Thakur's 1992 and 1996 articles use the variable q to represent the cardinality of a field. Note that $q = p^n$, where p is the characteristic of the field, and n is a positive integer. Since this thesis works with prime q, usually equal to 2 or 3, the terms "cardinality" and "characteristic" can be used interchangeably.

The fact that $q = p^n$ can be seen from the fact stated here:

> For F a finite field, there is a prime p and a positive integer n such that F has p^n elements. The prime number p is the characteristic of the field. (Zwillinger, 1996, p. 144)

B.4 Characteristic of a Field

Two definitions of "characteristic" are listed here.

> The *characteristic* of a field is the smallest positive integer n such that $1 + 1 + \cdots + 1 = 0$ (n summands). If no such n exists, the field has characteristic 0 (or characteristic ∞). (Zwillinger, 1996, p. 144)

> Characteristic (Field)
>
> For a FIELD K with multiplicative identity 1, consider the numbers $2 = 1 + 1$, $3 = 1 + 1 + 1$, $4 = 1 + 1 + 1 + 1$, etc. Either these numbers are all different, in which case we say that K has characteristic 0, or two of them will be equal. In this case, it is straightforward to show that, for some number p, we have $\underbrace{1 + 1 + \cdots + 1}_{p\,times} = 0$.
>
> If p is chosen to be as small as possible, then p will be a PRIME, and we say that K has characteristic p. The FIELDS $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and the p-ADIC NUMBERS $\mathbb{Q}_p$, have characteristic 0. For p, a PRIME, the GALOIS FIELD GF(p^n) has characteristic p.
>
> If H is a SUBFIELD of K, then H and K have the same characteristic. (Weisstein, 1999, p. 228)

B.5 Extension Field

A field E is an *extension field* of a field F if $F \subseteq E$ and the operations of F are those of E restricted to F. (Gallian, 2002, p. 344)

Example: $\mathbb{Z}_3[x]/\langle x^2+1\rangle$ is an extension field of $\mathbb{Z}_3$.

$\mathbb{Z}_3[x]$ is a ring of polynomials over a field defined as follows:

$\mathbb{Z}_3[x] = a_m x^n + a_{m-1}x^{n-1} + \cdots\ a_1$x + a_0, where $a_i \in \mathbb{Z}_3$, $0 \leq$ i $\leq$ m.

The elements of the quotient $\mathbb{Z}_3[x]/\langle x^2+1\rangle$ are the equivalence classes of {0, 1, 2, x, 2x, 1 + x, 1 + 2x, 2 + x, 2 + 2x}, where x can be thought of as i, since the class of x squared (x^2) is the class of -1. Note also that 2x can be thought of as $2i$.

You can represent each of these classes as above, since $[x]^2 = [-1]$ (or 2 in modulo 3 arithmetic) in $\mathbb{Z}_3[x]/\langle x^2+1\rangle$, where $[x]$ represents the class of x, $[-1]$ the class of -1.

Joseph Gallian discusses this field with Example 11, Chapter 17, p. 303, and Example 12, Chapter 14, p. 257, of "Contemporary Abstract Algebra."

B.6 Field

A *field* is a commutative ring with unity in which every nonzero element is a unit. (Gallian, 2002, p. 242)

Example: Field with Nine Elements

Let $\mathbb{Z}_3[i] = \{$a + bi | a, b, $\in \mathbb{Z}_3\}$
$= \{0, 1, 2, i, 1 + i, 2 + i, 1 + 2i, 2 + 2i\}$,
where $i^2 = -1$. This is the ring of Gaussian integers modulo 3. Elements are added and multiplied as in the complex numbers, except that the coefficients are reduced modulo 3. In particular, $-1 = 2$. (Gallian, 2002, p. 243)

Note that the cardinality of this field is 9, but that the characteristic is 3.

B.7 Field of Quotients

Let D be an integral domain. Then there exists a field F (called the field of quotients of D) that contains a subring isomorphic to D. (Gallian, 2002, p. 276)

B.8 Function Field $\mathbb{F}_q(t)$

Courtesy of Dr. Lee Rudolph, sci.net newsgroup, 2006 0528 post.

Let $\mathbb{F}_q[t]$ denote the polynomial ring over $\mathbb{F}_q$ in the single variable t: specifically, we could define $\mathbb{F}_q[t]$ to be that subset of the set of all sequences p: $N \rightarrow \mathbb{F}_q$ (where N contains 0) that are "eventually 0," with standard definitions of addition and multiplication.

Then $\mathbb{F}_q(t)$ is the quotient field of $\mathbb{F}_q[t]$: specifically, we could define $\mathbb{F}_q(t)$ to be the quotient set of the set of all pairs (p,q) in $\mathbb{F}_q[t] \times \mathbb{F}_q[t]$ such that q is not the identically-0 sequence, modulo the equivalence relation $\sim$ such that $(p,q) \sim (r,s)$ if and only if $ps = qr$.

B.9 Polynomial Ring over R

Let R be a commutative ring. The set of formal symbols

$R[x] = \{a_n x^n + a_{n-1}x^{n-1} + \cdots\ a_1\text{x} + a_0 \mid a_i \in \text{R, n is a nonnegative integer}\}$

is called the *ring of polynomials over R in the indeterminate x*. (Gallian, 2002, p. 283)

B.10 Ring

A *ring* is a nonempty set with two binary operations, addition (denoted by $a + b$) and multiplication (denoted by ab), such that for all a, b, c in R:

1. $a + b = b + a$

2. $(a + b) + c = a + (b + c)$.
3. There is an additive identity 0. That is, there is an element 0 in R such that $a + 0 = a$ for all a in R.
4. There is an element $-a$ in R such that $a + (-a) = 0$.
5. $a(bc) = (ab)c$.
6. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

So, a ring is an Abelian group under addition, also having an associative multiplication that is left and right distributive over addition. Note that multiplication need not be commutative. When it is, we say that the ring is *commutative*. Also, a ring need not have an identity under multiplication. A *unity* (or *identity*) in a ring is a nonzero element that is an identity under multiplication. A nonzero element of a commutative ring with unity need not have a multiplicative inverse. When it does, we say that is a *unit* of the ring. Thus, a is a unit if a^{-1} exists. (Gallian, 2002, ps. 229, 230))

Example: Ring with Two Elements

The set $\mathbb{Z}$ of integers under ordinary addition and multiplication is a commutative ring with unity 1. The units of $\mathbb{Z}$ are 1 and -1. (Gallian, 2002, p. 230)

B.11 Unit Group

Let R be a commutative ring with unity and let $U(R)$ denote the set of units of R. ...$U(R)$ is a group under the multiplication of R. (This group is called the *group of units of R*.) (Gallian, 2002, p. 236)

Appendix C

EXPANDED NOTATION

New "expanded notation" will now be introduced. Its purpose is to render continued fraction expansions of polynomials in t concise and easy to read. To this end, just the exponents of the terms in t will be listed in sequence. The variable t and the plus signs will be dropped from the representation.

For example, $t^2 + t^3 + t^4$ will be denoted as $\{2, 3, 4\}$.

C.1 "Curly Bracket" Notation for $q = 2$

When cardinality $q = 2$, "Curly Bracket" notation uses curly brackets, "{" and "}" to enclose just the exponents of the polynomial in t for each partial quotient of the continued fraction expansion. The correlation is as follows:

Table C.1: "Curly Bracket" Notation Correlation, $q = 2$

$0 \rightarrow 0$	$1 \rightarrow 1$ or $\{0\}$	$t \rightarrow \{1\}$
$1 + t \rightarrow \{0, 1\}$	$t^2 \rightarrow \{2\}$	$1 + t^2 \rightarrow \{0, 2\}$
$t + t^2 \rightarrow \{1, 2\}$	$1 + t + t^2 \rightarrow \{0, 1, 2\}$	$t^3 \rightarrow \{3\}$, etc.

So,

$[0, t + t^2, t + t^2 + t^3 + t^4 + t^5 + t^6, t + t^2, 1 + t^2 + t^3 + t^5 + t^6, t + t^4, t + t^2, t + t^4]$

maps to

$[0, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{0, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\}]$

C.2 "Colon" Notation for $q = 3$

When cardinality $q = 3$, "Colon" notation uses a colon, ":" to separate the coefficient and exponent of each term of a polynomial in t for each partial quotient of the continued fraction expansion. It also uses curly brackets, as seen with $q = 2$, to separate the partial quotients of the expansion. The correlation is as follows:

Table C.2: "Colon" Notation Correlation, $q = 3$

$0 \to 0$ or $\{0{:}0\}$	$1 \to 1$ or $\{1{:}0\}$	$2 \to 2$ or $\{2{:}0\}$
$t \to \{1{:}1\}$	$2t \to \{2{:}1\}$	$1 + t \to \{1{:}0,\ 1{:}1\}$
$2 + t \to \{2{:}0,\ 1{:}1\}$	$1 + 2t \to \{1{:}0,\ 2{:}1\}$	$2 + 2t \to \{2{:}0,\ 2{:}1\}$
$t^2 \to \{1{:}2\}$	$2t^2 \to \{2{:}2\}$	$1 + t^2 \to \{1{:}0,\ 1{:}2\}$
$2 + t^2 \to \{2{:}0,\ 1{:}2\}$	$1 + 2t^2 \to \{1{:}0,\ 2{:}2\}$	$2 + 2t^2 \to \{2{:}0,\ 2{:}2\}$, etc.

So,

$[2t^4 + t^6, 2t^5 + t^7 + t^{13} + 2t^{15}, t^4 + 2t^6, t^{14} + 2t^{20} + 2t^{22} + t^{28} + 2t^{40} + t^{46} + t^{48} + 2t^{54}]$

maps to

[{2:4, 1:6}, {2:5, 1:7, 1:13, 2:15}, {1:4, 2:6}, {1:14, 2:20, 2:22, 1:28, 2:40, 1:46, 1:48, 2:54}]

Using "Colon" notation is relevant for all cardinalities $q > 2$.

APPENDIX D

SEQUENCE EXPANSIONS

This appendix includes expansions of several numbers mentioned in this thesis. This includes an extended expansion of $e(1)$ for cardinality $q = 2$, and several expansions of general Hurwitz numbers in the real number setting. Also included is a pattern and sign alternation sequence expansion, which relates to the $e(z)$ function for all cardinalities.

D.1 Theorem 1. $e(1)$ to 513 Places, $q = 2$

Chapter 2.2.2, pg. 12, Theorem 1. $e(1)$ to 513 places, $q = 2$

$e(1)$ = [1, [1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [6],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [7],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [6],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [8],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [6],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [7],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [6],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [9],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],

[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [6],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [7],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [6],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [8],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [6],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [7],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [6],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [10], ...].

D.2 Pattern and Sign Alternation

Included here is an expansion of the exponential function, as it relates to the placement of the terms, and the sign value of the terms. This pattern will be applicable for all cardinalities. Note that with cardinality $q = 2$, all the signs will be positive.

Note that if you take each a_n in turn, and follow its sign alternation throughout the expansion, the signs exactly alternate between positive and negative. For example, you see [..., a_2, ..., $-a_2$, ..., a_2, ..., $-a_2$, ...].

Pattern of $e(z) =$

[a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, a_5,
a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, -a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, a_6,
a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, -a_5,
a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, -a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, a_7,

a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, a_5,

a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, -a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, -a_6,

a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, -a_5,

a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, -a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, a_8,

a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, a_5,

a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, -a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, a_6,

a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, -a_5,

a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, -a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, -a_7,

a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, a_5,

a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, -a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, -a_6,

a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, -a_5,

a_1, a_2, -a_1, a_3, a_1, -a_2, -a_1, -a_4, a_1, a_2, -a_1, -a_3, a_1, -a_2, -a_1, a_9, ...]

D.3 $(ae^{2/n} + b)/(ce^{2/n} + d)$

The following are examples of general Hurwitz numbers over the real numbers, $\mathbb{R}$. This was discussed in Chapter 3.1.2, pg. 24.

D.3.1 $(e + 2)/(3e + 4)$

[0, 2, 1, 1, 2, 1, 3, 1, 1, 1, 3, 3, 3, 1, 3, 1, 3, 5, 3, 1, 5, 1, 3, 7, 3, 1, 7, 1, 3, 9, 3, 1, 9, 1, 3, 11, 3, 1, 11, 1, 3, 13, 3, 1, 13, 1, 3, 15, 3, 1, 15, 1, 3, 17, 3, 1, 17, 1, 3, 19, 3, 1, 19, 1, 3, 21, 3, 1, 21, 1, 3, 23, 3, 1, 23, 1, 3, 25, 3, 1, 25, 1, 3, 27, 3, 1, 27, 1, 3, 29, 3, 1, 29, 1, 3, 31, 3, 1, 31, 1, ...]

D.3.2 $(2e + 3)/(4e + 5)$

[0, 1, 1, 7, 2, 3, 2, 3, 1, 2, 1, 3, 4, 3, 1, 4, 1, 3, 6, 3, 1, 6, 1, 3, 8, 3, 1, 8, 1, 3, 10, 3, 1, 10, 1, 3, 12, 3, 1, 12, 1, 3, 14, 3, 1, 14, 1, 3, 16, 3, 1, 16, 1, 3, 18, 3, 1, 18, 1, 3, 20, 3, 1, 20, 1, 3, 22, 3, 1, 22, 1, 3, 24, 3, 1, 24, 1, 3, 26, 3, 1, 26, 1, 3, 28, 3, 1, 28, 1, 3, 30, 3, 1, 30, 1, 3, 32, 3, 1, 32, ...]

D.3.3 $(e^{2/3}+2)/(3e^{2/3}+4)$

[0, 2, 2, 37, 3, 1, 3, 5, 108, 8, 3, 1, 9, 180, 12, 1, 3, 14, 252, 17, 3, 1, 18, 324, 21, 1, 3, 23, 396, 26, 3, 1, 27, 468, 30, 1, 3, 32, 540, 35, 3, 1, 36, 612, 39, 1, 3, 41, 684, 44, 3, 1, 45, 756, 48, 1, 3, 50, 828, 53, 3, 1, 54, 900, 57, 1, 3, 59, 972, 62, 3, 1, 63, 1044, 66, 1, 3, 68, 1116, 71, 3, 1, 72, 1188, 75, 1, 3, 77, 1260, 80, 3, 1, 81, 1332, 84, 1, 3, 86, 1404, 89, ...]

D.3.4 $(2e^{2/3}+3)/(4e^{2/3}+5)$

[0, 1, 1, 5, 1, 8, 1, 1, 3, 3, 1, 4, 1, 1, 26, 1, 1, 7, 1, 3, 9, 1, 1, 44, 1, 1, 12, 3, 1, 13, 1, 1, 62, 1, 1, 16, 1, 3, 18, 1, 1, 80, 1, 1, 21, 3, 1, 22, 1, 1, 98, 1, 1, 25, 1, 3, 27, 1, 1, 116, 1, 1, 30, 3, 1, 31, 1, 1, 134, 1, 1, 34, 1, 3, 36, 1, 1, 152, 1, 1, 39, 3, 1, 40, 1, 1, 170, 1, 1, 43, 1, 3, 45, 1, 1, 188, 1, 1, 48, 3, ...]

D.3.5 $(e^{2/5}+2)/(3e^{2/5}+4)$

[0, 2, 2, 2, 1, 14, 1, 1, 5, 1, 3, 8, 1, 1, 44, 1, 1, 13, 3, 1, 15, 1, 1, 74, 1, 1, 20, 1, 3, 23, 1, 1, 104, 1, 1, 28, 3, 1, 30, 1, 1, 134, 1, 1, 35, 1, 3, 38, 1, 1, 164, 1, 1, 43, 3, 1, 45, 1, 1, 194, 1, 1, 50, 1, 3, 53, 1, 1, 224, 1, 1, 58, 3, 1, 60, 1, 1, 254, 1, 1, 65, 1, 3, 68, 1, 1, 284, 1, 1, 73, 3, 1, 75, 1, 1, 314, 1, 1, 80, 1, ...]

D.3.6 $(2e^{2/5}+3)/(4e^{2/5}+5)$

[0, 1, 1, 4, 1, 60, 6, 3, 1, 8, 180, 13, 1, 3, 16, 300, 21, 3, 1, 23, 420, 28, 1, 3, 31, 540, 36, 3, 1, 38, 660, 43, 1, 3, 46, 780, 51, 3, 1, 53, 900, 58, 1, 3, 61, 1020, 66, 3, 1, 68, 1140, 73, 1, 3, 76, 1260, 81, 3, 1, 83, 1380, 88, 1, 3, 91, 1500, 96, 3, 1, 98, 1620, 103, 1, 3, 106, 1740, 111, 3, 1, 113, 1860, 118, 1, 3, 121, 1980, 126, 3, 1, 128, 2100, 133, 1, 3, 136, 2220, 141, 3, 1, 143, ...]

Appendix E

HURWITZ NUMBERS AND DEVELOPMENT

This appendix lists many simple and general Hurwitz number continued fraction sequence expansions. The expansions are prefaced with a section describing the detailed mathematical process of calculating these sequences. The “Rational Function to Simple CF Expansion Algorithm,” the essential core of this process, is explained in detail.

E.1 Algorithm to Calculate Continued Fractions for Real Rational Numbers

Prior to discussing Hurwitz Number evaluation, look quickly at the algorithm for evaluating the continued fraction expansion of a rational real number.

Look at the fraction $\frac{101}{1001}$. To evaluate its continued fraction expansion, first calculate its integer quotient and the reciprocal of the remainder. Then repeat this calculation, starting with the reciprocal of the remainder each time. Stop when the remainder is 0. Finally, construct the continued fraction expansion of the fraction using each integer quotient as a successive a_i term, as described in Chapter 2.2.1, p. 9.

For $\frac{101}{1001}$, the integer quotient is 0, the remainder is $\frac{101}{1001}$, and the reciprocal of the remainder is $\frac{1001}{101}$.

For $\frac{1001}{101}$, the integer quotient is 9, the remainder is $\frac{92}{101}$, and the reciprocal of the remainder is $\frac{101}{92}$.

For $\frac{101}{92}$, the integer quotient is 1, the remainder is $\frac{9}{92}$, and the reciprocal of the remainder is $\frac{92}{9}$.

For $\frac{92}{9}$, the integer quotient is 10, the remainder is $\frac{2}{9}$, and the reciprocal of the remainder is $\frac{9}{2}$.

For $\frac{9}{2}$, the integer quotient is 4, the remainder is $\frac{1}{2}$, and the reciprocal of the remainder is 2. Since 2 is an integer, stop here.

Finally, the continued fraction expansion of $\frac{101}{1001}$ is [0, 9, 1, 10, 4, 2].

E.2 Process for Simple Hurwitz Number Evaluation

Evaluating simple Hurwitz numbers of the form:

$$(\tfrac{x}{y})e(\tfrac{\theta}{f}) + (\tfrac{z}{w})$$

with $x,\ y,\ z,\ w,\ f \in \mathbb{F}_q[t]$, $f \neq 0$, $xw - yz \neq 0$ and $\theta \in \mathbb{F}_q^*$, is a three part process.

Because of the complexity of the CF expansion, a Mathematica 5.2 program has been written which does the following:

1. For a manageable number n, such as 7, evaluate $x_7 := \sum_{i=1}^{7} \frac{(\frac{\theta}{f})^{q^i}}{D_i}$. This process was explained in Theorem 1, Chapter 2.2.2, p. 12. Reduce x_7 by modulo q if appropriate. With Mathematica 5.2, this is done with the PolynomialMod[%, q] command.

2. Then, add the a_0 term, $\frac{(\frac{\theta}{f})}{1}$, to x_7. This gives you "$\frac{(\frac{\theta}{f})}{1} + \sum_{i=1}^{7} \frac{(\frac{\theta}{f})^{q^i}}{D_i}$." Then, multiply this sum by $\frac{x}{y}$ and add $\frac{z}{w}$ to the product.

   Then, the rational function form of $(\frac{x}{y})e(\frac{\theta}{f}) + (\frac{z}{w})$ is approximately:
   $(\frac{x}{y})(\frac{(\frac{\theta}{f})}{1} + \sum_{i=1}^{7} \frac{(\frac{\theta}{f})^{q^i}}{D_i}) + (\frac{z}{w})$. Reduce this new expression by modulo q if appropriate.

3. *Rational Function to Simple CF Expansion Algorithm*

   Convert the rational function form of the existing expression to its simple CF form. Do this by calculating the polynomial quotient, partial quotient by partial quotient, and carrying over the remainder and denominator with each term, as follows:

a) Evaluate $(\frac{x}{y})(\frac{(\frac{\theta}{f})}{1} + \sum_{i=1}^{7} \frac{(\frac{\theta}{f})^{q^i}}{D_i}) + (\frac{z}{w})$. Define pq_0 to be the polynomial quotient of this first expression obtained with the division algorithm. Define pr_0 to be the polynomial remainder and pd_0 to be the denominator of this first expression. Place pq_0 in the first position of a new array and pass pr_0 and pd_0 on to the second step.

b) Evaluate $(\frac{pd_0}{pr_0})$. Reduce this new expression by modulo q if appropriate. Define pq_1 to be the polynomial quotient of this second expression. Define pr_1 to be the polynomial remainder and pd_1 to be the denominator of this second expression. Place pq_1 in the second position of the new array and pass pr_1 and pd_1 on to the third step.

c) Evaluate $(\frac{pd_1}{pr_1})$. Reduce this new expression by modulo q if appropriate. Define pq_2 to be the polynomial quotient of this third expression. Define pr_2 to be the polynomial remainder and pd_2 to be the denominator of this third expression. Place pq_2 in the third position of the new array and pass pr_2 and pd_2 on to the fourth step.

d) Repeat this process as many times as desired to explore and understand the expansion for the specified Hurwitz number. The new array, to which the pq_i, i ranging from zero to infinity, expressions have been added, will be the simple Hurwitz number CF expansion, or simple CF expansion.

The Mathematica 5.2 program for this algorithm can be found in Appendix I, p. 113.

E.3 Evaluations of $e(z) - z$ for Various q, z

E.3.1 $e(1) + 1$ for $q = 2$

The following is an expansion for $e(1) + 1$ for $q = 2$.

As shown with Theorem 1, 2.2.2, p. 12,

$e(1) = [1,$
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [6], ...]

$e(1) + 1 = [0,$
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [5],
[1], [2], [1], [3], [1], [2], [1], [4], [1], [2], [1], [3], [1], [2], [1], [6], ...]

With the "Curly Bracket" expanded notation described in Appendix C.1, p. 73, this expands to the following:

$e(1) + 1 = [0,$
{1, 2}, {1, 4}, {1, 2}, {1, 8}, {1, 2}, {1, 4}, {1, 2}, {1, 16},
{1, 2}, {1, 4}, {1, 2}, {1, 8}, {1, 2}, {1, 4}, {1, 2}, {1, 32},
{1, 2}, {1, 4}, {1, 2}, {1, 8}, {1, 2}, {1, 4}, {1, 2}, {1, 16},
{1, 2}, {1, 4}, {1, 2}, {1, 8}, {1, 2}, {1, 4}, {1, 2}, {1, 64}, ...]

E.3.2 $e(t) + t$ for $q = 2$

The following is $e(t) + t$ for $q = 2$.

The first 65 terms of the simple CF expansion are the following:

$e(t) + t =$ [1, {0, 1}, {2}, {1, 2}, {6},
{0, 1}, {1, 4}, {0, 1}, {14}, {1, 2}, {2},
{0, 1}, {1, 8}, {0, 1}, {2}, {1, 2}, {30},
{0, 1}, {1, 4}, {0, 1}, {6}, {1, 2}, {2},
{0, 1}, {1, 16}, {0, 1}, {2}, {1, 2}, {6},
{0, 1}, {1, 4}, {0, 1}, {62}, {1, 2}, {2},
{0, 1}, {1, 8}, {0, 1}, {2}, {1, 2}, {14},

{0, 1}, {1, 4}, {0, 1}, {6}, {1, 2}, {2},
{0, 1}, {1, 32}, {0, 1}, {2}, {1, 2}, {6},
{0, 1}, {1, 4}, {0, 1}, {14}, {1, 2}, {2},
{0, 1}, {1, 8}, {0, 1}, {2}, {1, 2}, {126}, ...]

Already, the pattern has become more interesting.

Again, note that "Curly Bracket" notation defined in Appendix C.1 on page 73 is used.

E.3.3 $e(\frac{1}{t}) + \frac{1}{t}$ *for* $q = 2$

The following is $e(\frac{1}{t}) + \frac{1}{t}$ for $q = 2$.

The first 65 terms of the simple CF expansion:

$e(\frac{1}{t}) + \frac{1}{t} =$ [0,
{3, 4}, {1, 4}, {3, 4}, {1, 8}, {3, 4}, {1, 4}, {3, 4}, {1, 16},
{3, 4}, {1, 4}, {3, 4}, {1, 8}, {3, 4}, {1, 4}, {3, 4}, {1, 32},
{3, 4}, {1, 4}, {3, 4}, {1, 8}, {3, 4}, {1, 4}, {3, 4}, {1, 16},
{3, 4}, {1, 4}, {3, 4}, {1, 8}, {3, 4}, {1, 4}, {3, 4}, {1, 64},
{3, 4}, {1, 4}, {3, 4}, {1, 8}, {3, 4}, {1, 4}, {3, 4}, {1, 16},
{3, 4}, {1, 4}, {3, 4}, {1, 8}, {3, 4}, {1, 4}, {3, 4}, {1, 32},
{3, 4}, {1, 4}, {3, 4}, {1, 8}, {3, 4}, {1, 4}, {3, 4}, {1, 16},
{3, 4}, {1, 4}, {3, 4}, {1, 8}, {3, 4}, {1, 4}, {3, 4}, {1, 28}, ...]

E.3.4 $e(\frac{t+1}{t}) + \frac{t+1}{t}$ *for* $q = 2$

The following is $e(\frac{t+1}{t}) + \frac{t+1}{t}$ for $q = 2$.

The first 65 terms of the simple CF expansion:

$e(\frac{t+1}{t}) + \frac{t+1}{t} =$ [0, {0, 1, 2}, {0, 1}, {0, 2}, {3, 4},
{0, 2, 4, 6},
{0, 1}, {0, 1, 2}, {1, 4}, {0, 1, 2}, {0, 1},
{0, 2, 4, 6, 8, 10, 12, 14},
{3, 4}, {0, 2}, {0, 1}, {0, 1, 2}, {1, 8}, {0, 1, 2}, {0, 1}, {0, 2}, {3, 4},

{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30},

{0, 1}, {0, 1, 2}, {1, 4}, {0, 1, 2}, {0, 1},

{0, 2, 4, 6},

{3, 4}, {0, 2}, {0, 1}, {0, 1, 2}, {1, 16}, {0, 1, 2}, {0, 1}, {0, 2}, {3, 4},

{0, 2, 4, 6},

{0, 1}, {0, 1, 2}, {1, 4}, {0, 1, 2}, {0, 1},

{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62},

{3, 4}, {0, 2}, {0, 1}, {0, 1, 2}, {1, 8}, {0, 1, 2}, {0, 1}, {0, 2}, {3, 4},

{0, 2, 4, 6, 8, 10, 12, 14},

{0, 1}, {0, 1, 2}, {1, 4}, {0, 1, 2}, {0, 1},

{0, 2, 4, 6},

{3, 4}, {0, 2}, {0, 1}, {0, 1, 2}, {1, 32}, ...]

E.3.5 $e(1) - 1$ for $q = 3$

Now, examine $e(1) - 1$ for $q = 3$.

With the "Colon" expanded notation described in Appendix C.2, p. 74, the first 60 terms of the simple CF expansion expand to the following:

$e(1) - 1$ = [0, {2:1, 1:3}, {2:2, 1:4, 1:10, 2:12}, {1:1, 2:3},

{1:5, 2:11, 2:13, 1:19, 2:31, 1:37, 1:39, 2:45},

{2:1, 1:3}, {1:2, 2:4, 2:10, 1:12}, {1:1, 2:3},

{2:14, 1:32, 1:38, 1:40, 2:56, 2:58, 2:64, 1:82, 1:94, 2:112, 2:118, 2:120, 1:136, 1:138, 1:144, 2:162},

{2:1, 1:3}, {2:2, 1:4, 1:10, 2:12}, {1:1, 2:3},

{2:5, 1:11, 1:13, 2:19, 1:31, 2:37, 2:39, 1:45},

{2:1, 1:3}, {1:2, 2:4, 2:10, 1:12}, {1:1, 2:3},

{1:41, 2:95, 2:113, 2:119, 2:121, 1:167, 1:173, 1:175, 1:191, 1:193, 1:199, 2:245, 2:247, 2:253, 2:271, 2:283, 1:325, 1:337, 1:355, 1:361, 1:363, 2:409, 2:415, 2:417, 2:433, 2:435, 2:441, 1:487, 1:489, 1:495, 1:513, 2:567},

{2:1, 1:3}, {2:2, 1:4, 1:10, 2:12}, {1:1, 2:3},

{1:5, 2:11, 2:13, 1:19, 2:31, 1:37, 1:39, 2:45},

{2:1, 1:3}, {1:2, 2:4, 2:10, 1:12}, {1:1, 2:3},

{1:14, 2:32, 2:38, 2:40, 1:56, 1:58, 1:64, 2:82, 2:94, 1:112, 1:118, 1:120, 2:136, 2:138, 2:144, 1:162},

{2:1, 1:3}, {2:2, 1:4, 1:10, 2:12}, {1:1, 2:3},

{2:5, 1:11, 1:13, 2:19, 1:31, 2:37, 2:39, 1:45},

{2:1, 1:3}, {1:2, 2:4, 2:10, 1:12}, {1:1, 2:3},

{2:122, 1:284, 1:338, 1:356, 1:362, 1:364, 2:500, 2:518, 2:524, 2:526, 2:572, 2:578, 2:580, 2:596, 2:598, 2:604, 1:734, 1:740, 1:742, 1:758, 1:760, 1:766, 1:812, 1:814, 1:820, 1:838, 1:850, 2:974, 2:976, 2:982, 2:1000, 2:1012, 2:1054, 2:1066, 2:1084, 2:1090, 2:1092, 1:1216, 1:1228, 1:1246, 1:1252, 1:1254, 1:1300, 1:1306, 1:1308, 1:1324, 1:1326, 1:1332, 2:1462, 2:1468, 2:1470, 2:1486, 2:1488, 2:1494, 2:1540, 2:1542, 2:1548, 2:1566, 1:1702, 1:1704, 1:1710, 1:1728, 1:1782, 2:1944},

{2:1, 1:3}, {2:2, 1:4, 1:10, 2:12}, {1:1, 2:3},

{1:5, 2:11, 2:13, 1:19, 2:31, 1:37, 1:39, 2:45},

{2:1, 1:3}, {1:2, 2:4, 2:10, 1:12}, {1:1, 2:3},

{2:14, 1:32, 1:38, 1:40, 2:56, 2:58, 2:64, 1:82, 1:94, 2:112, 2:118, 2:120, 1:136, 1:138, 1:144, 2:162},

{2:1, 1:3}, {2:2, 1:4, 1:10, 2:12}, {1:1, 2:3},

{2:5, 1:11, 1:13, 2:19, 1:31, 2:37, 2:39, 1:45},

{2:1, 1:3}, {1:2, 2:4, 2:10, 1:12}, {1:1, 2:3},

{2:41, 1:95, 1:113, 1:119, 1:121, 2:167, 2:173, 2:175, 2:191, 2:193, 2:199, 1:245, 1:247, 1:253, 1:271, 1:283, 2:325, 2:337, 2:355, 2:361, 2:363, 1:409, 1:415, 1:417, 1:433, 1:435, 1:441, 2:487, 2:489, 2:495, 2:513, 1:567},

{2:1, 1:3}, {2:2, 1:4, 1:10, 2:12}, {1:1, 2:3},

{1:5, 2:11, 2:13, 1:19, 2:31, 1:37, 1:39, 2:45},

{2:1, 1:3}, {1:2, 2:4, 2:10, 1:12}, {1:1, 2:3},

{1:14, 2:32, 2:38, 2:40, 1:56, 1:58, 1:64, 2:82, 2:94, 1:112, 1:118, 1:120, 2:136, 2:138, 2:144, 1:162},

{2:1, 1:3}, {2:2, 1:4, 1:10, 2:12}, {1:1, 2:3}, ...]

E.3.6 $e(t) - t$ *for* $q = 3$

The following is $e(t) - t$ for $q = 3$.

The first 60 terms of the simple CF expansion:

$e(t) - t =$ [1, {2:0, 1:2}, {1:3, 2:5}, {2:1, 2:2}, {2:0, 2:1}, {2:0, 2:1}, {1:1}, {1:0, 2:2},

{1:0, 2:12, 1:18, 1:20, 2:26},

{2:0, 1:2}, {2:0, 2:1}, {1:0, 2:1, 1:2}, {2:2, 1:3}, {1:2},

{1:0, 2:1, 1:2, 2:3, 1:4},

{1:0, 1:1}, {1:0, 1:1},

{2:1, 2:2, 1:3, 1:4, 2:8, 1:9, 1:10, 2:11},

{2:0, 2:1},

{1:1, 1:3, 1:9, 2:27, 2:39, 1:57, 1:63, 1:65, 2:81, 2:83, 2:89, 1:107},

{1:0, 1:1},

{2:1, 2:2, 2:4, 2:5, 1:6, 1:7, 1:8, 1:9, 2:10, 1:11},

{2:1}, {2:0, 1:1, 2:3}, {2:1, 1:3, 2:5}, {1:1}, {2:1}, {1:0, 1:1}, {2:0, 2:1}, {1:1, 2:2}, {1:0, 1:1, 1:2, 2:3}, {1:1}, {1:1}, {1:0, 2:1, 2:2}, {2:0, 1:1, 1:2, 1:3}, {2:2}, {2:0, 2:1}, {2:0, 2:1}, {1:0, 2:1, 2:3}, {2:1}, {2:1}, {2:0, 1:1}, {2:1}, {1:0, 1:1}, {2:1}, {2:1}, {2:0, 1:2}, {2:0, 1:1}, {2:0, 1:1},

{1:0, 2:1, 1:2, 2:3, 1:4, 2:5, 1:6},

{1:0, 1:1}, {1:0, 2:1, 1:2}, {2:0, 2:1}, {1:0, 1:1}, {1:0, 2:1}, {1:0, 2:1, 1:2},

{1:0, 2:1, 1:2, 1:3, 1:4},

{1:0, 2:1}, {2:0, 1:1}, {1:0, 2:1}, ...]

Again, note that “Colon” notation defined in Appendix C.2 on page 74 is used.

E.3.7 $e(\frac{1}{t}) - \frac{1}{t}$ *for* $q = 3$

The following is $e(\frac{1}{t}) - \frac{1}{t}$ for $q = 3$.

The first 60 terms of the simple CF expansion:

$e(\frac{1}{t}) - \frac{1}{t} =$ [0, {2:4, 1:6}, {2:5, 1:7, 1:13, 2:15}, {1:4, 2:6},

{1:14, 2:20, 2:22, 1:28, 2:40, 1:46, 1:48, 2:54},

{2:4, 1:6}, {1:5, 2:7, 2:13, 1:15}, {1:4, 2:6},

{2:41, 1:59, 1:65, 1:67, 2:83, 2:85, 2:91, 1:109, 1:121, 2:139, 2:145, 2:147, 1:163, 1:165, 1:171, 2:189},

{2:4, 1:6}, {2:5, 1:7, 1:13, 2:15}, {1:4, 2:6},

{2:14, 1:20, 1:22, 2:28, 1:40, 2:46, 2:48, 1:54},

{2:4, 1:6}, {1:5, 2:7, 2:13, 1:15}, {1:4, 2:6},

{1:122, 2:176, 2:194, 2:200, 2:202, 1:248, 1:254, 1:256, 1:272, 1:274, 1:280, 2:326, 2:328, 2:334, 2:352, 2:364, 1:406, 1:418, 1:436, 1:442, 1:444, 2:490, 2:496, 2:498, 2:514, 2:516, 2:522, 1:568, 1:570, 1:576, 1:594, 2:648},

{2:4, 1:6}, {2:5, 1:7, 1:13, 2:15}, {1:4, 2:6},

{1:14, 2:20, 2:22, 1:28, 2:40, 1:46, 1:48, 2:54},

{2:4, 1:6}, {1:5, 2:7, 2:13, 1:15}, {1:4, 2:6},

{1:41, 2:59, 2:65, 2:67, 1:83, 1:85, 1:91, 2:109, 2:121, 1:139, 1:145, 1:147, 2:163, 2:165, 2:171, 1:189},

{2:4, 1:6}, {2:5, 1:7, 1:13, 2:15}, {1:4, 2:6},

{2:14, 1:20, 1:22, 2:28, 1:40, 2:46, 2:48, 1:54},

{2:4, 1:6}, {1:5, 2:7, 2:13, 1:15}, {1:4, 2:6},

{2:365, 1:527, 1:581, 1:599, 1:605, 1:607, 2:743, 2:761, 2:767, 2:769, 2:815, 2:821, 2:823, 2:839, 2:841, 2:847, 1:977, 1:983, 1:985, 1:1001, 1:1003, 1:1009, 1:1055, 1:1057, 1:1063, 1:1081, 1:1093, 2:1217, 2:1219, 2:1225, 2:1243, 2:1255, 2:1297, 2:1309, 2:1327, 2:1333, 2:1335, 1:1459, 1:1471, 1:1489, 1:1495, 1:1497, 1:1543, 1:1549, 1:1551, 1:1567, 1:1569, 1:1575, 2:1705, 2:1711, 2:1713, 2:1729, 2:1731, 2:1737, 2:1783, 2:1785, 2:1791, 2:1809, 1:1945, 1:1947, 1:1953, 1:1971, 1:2025, 2:2187},

{2:4, 1:6}, {2:5, 1:7, 1:13, 2:15}, {1:4, 2:6},

{1:14, 2:20, 2:22, 1:28, 2:40, 1:46, 1:48, 2:54},

{2:4, 1:6}, {1:5, 2:7, 2:13, 1:15}, {1:4, 2:6},

{2:41, 1:59, 1:65, 1:67, 2:83, 2:85, 2:91, 1:109, 1:121, 2:139, 2:145, 2:147, 1:163, 1:165, 1:171, 2:189},

{2:4, 1:6}, {2:5, 1:7, 1:13, 2:15}, {1:4, 2:6},

{2:14, 1:20, 1:22, 2:28, 1:40, 2:46, 2:48, 1:54},

{2:4, 1:6}, {1:5, 2:7, 2:13, 1:15}, {1:4, 2:6},

{2:122, 1:176, 1:194, 1:200, 1:202, 2:248, 2:254, 2:256, 2:272, 2:274, 2:280, 1:326, 1:328, 1:334, 1:352, 1:364, 2:406, 2:418, 2:436, 2:442, 2:444, 1:490, 1:496, 1:498, 1:514, 1:516, 1:522, 2:568, 2:570, 2:576, 2:594, 1:648},

{2:4, 1:6}, {2:5, 1:7, 1:13, 2:15}, {1:4, 2:6},

{1:14, 2:20, 2:22, 1:28, 2:40, 1:46, 1:48, 2:54},

{2:4, 1:6}, {1:5, 2:7, 2:13, 1:15}, {1:4, 2:6},

{1:41, 2:59, 2:65, 2:67, 1:83, 1:85, 1:91, 2:109, 2:121, 1:139, 1:145, 1:147, 2:163, 2:165, 2:171, 1:189},

{2:4, 1:6}, {2:5, 1:7, 1:13, 2:15}, {1:4, 2:6}, ...]

E.4 Evaluations of $xe(z) - xz$ for Various q, z, and x

From Chapter 2.1.3, p. 9, it is seen that the first few terms of $e(z)$ look like the following:

$$\begin{aligned} e(z) &= \tfrac{z}{1} + \tfrac{z^q}{D_1} + \tfrac{z^{q^2}}{D_2} + \tfrac{z^{q^3}}{D_3} + \tfrac{z^{q^4}}{D_4} + \cdots \\ &= \tfrac{z}{1} + \tfrac{z^q}{[1]} + \tfrac{z^{q^2}}{[2][1]^q} + \tfrac{z^{q^3}}{[3][2]^q[1]^{q^2}} + \tfrac{z^{q^4}}{[4][3]^q[2]^{q^2}[1]^{q^3}} + \cdots . \end{aligned}$$

So, accordingly, to determine $xe(z) - xz$, evaluate the CF expansion of

$$\begin{aligned} xe(z) - xz &= \tfrac{xz^q}{D_1} + \tfrac{xz^{q^2}}{D_2} + \tfrac{xz^{q^3}}{D_3} + \tfrac{xz^{q^4}}{D_4} + \cdots \\ &= \tfrac{xz^q}{[1]} + \tfrac{xz^{q^2}}{[2][1]^q} + \tfrac{xz^{q^3}}{[3][2]^q[1]^{q^2}} + \tfrac{xz^{q^4}}{[4][3]^q[2]^{q^2}[1]^{q^3}} + \cdots . \end{aligned}$$

E.4.1 $te(1) + t$ for $q = 2$

Now, examine $te(1) + t$ for $q = 2$.

Let $q = 2$, $z = 1$, and $x = t$.

$$\begin{aligned} te(1) + t &= \tfrac{t}{D_1} + \tfrac{t}{D_2} + \tfrac{t}{D_3} + \tfrac{t}{D_4} + \cdots \\ &= \tfrac{t}{[1]} + \tfrac{t}{[2][1]^2} + \tfrac{t}{[3][2]^2[1]^4} + \tfrac{t}{[4][3]^2[2]^4[1]^8} + \cdots . \end{aligned}$$

To expand $te(1) + t$, follow the simple Hurwitz number evaluation process outlined in the beginning of this Appendix, E.2, 80.

At Item 2 of the process, evaluate:

$$(\mathrm{t})(1 + \textstyle\sum_{i=1}^{7} \tfrac{1}{D_i}) + (\mathrm{t})$$

Simple CF expansion:

$te(1) + t = [0,$
$\{0, 1\}, \{2, 5\}, \{0, 1\}, \{2, 9\}, \{0, 1\}, \{2, 5\}, \{0, 1\}, \{2, 17\},$
$\{0, 1\}, \{2, 5\}, \{0, 1\}, \{2, 9\}, \{0, 1\}, \{2, 5\}, \{0, 1\}, \{2, 33\},$

{0, 1}, {2, 5}, {0, 1}, {2, 9}, {0, 1}, {2, 5}, {0, 1}, {2, 17},
{0, 1}, {2, 5}, {0, 1}, {2, 9}, {0, 1}, {2, 5}, {0, 1}, {2, 65},
{0, 1}, {2, 5}, {0, 1}, {2, 9}, {0, 1}, {2, 5}, {0, 1}, {2, 17},
{0, 1}, {2, 5}, {0, 1}, {2, 9}, {0, 1}, {2, 5}, {0, 1}, {2, 33},
{0, 1}, {2, 5}, {0, 1}, {2, 9}, {0, 1}, {2, 5}, {0, 1}, {2, 17},
{0, 1}, {2, 5}, {0, 1}, {2, 9}, {0, 1}, {2, 5}, {0, 1}, {2, 129}, ...]

E.4.2 $\frac{1}{t}e(1) + \frac{1}{t}$ *for* $q = 2$

The following is $\frac{1}{t}e(1) + \frac{1}{t}$ for $q = 2$.

Let $q = 2$, $z = 1$, and $x = \frac{1}{t}$.

$$\begin{aligned}\tfrac{1}{t}e(1) - \tfrac{1}{t} &= \tfrac{1}{tD_1} + \tfrac{1}{tD_2} + \tfrac{1}{tD_3} + \tfrac{1}{tD_4} + \cdots \\ &= \tfrac{1}{t[1]} + \tfrac{1}{t[2][1]^2} + \tfrac{1}{t[3][2]^2[1]^4} + \tfrac{1}{t[4][3]^2[2]^4[1]^8} + \cdots.\end{aligned}$$

Simple CF expansion:

$\frac{1}{t}e(1) + \frac{1}{t} = [0,$
{2, 3}, {0, 3}, {2, 3}, {0, 7}, {2, 3}, {0, 3}, {2, 3}, {0, 15},
{2, 3}, {0, 3}, {2, 3}, {0, 7}, {2, 3}, {0, 3}, {2, 3}, {0, 31},
{2, 3}, {0, 3}, {2, 3}, {0, 7}, {2, 3}, {0, 3}, {2, 3}, {0, 15},
{2, 3}, {0, 3}, {2, 3}, {0, 7}, {2, 3}, {0, 3}, {2, 3}, {0, 63},
{2, 3}, {0, 3}, {2, 3}, {0, 7}, {2, 3}, {0, 3}, {2, 3}, {0, 15},
{2, 3}, {0, 3}, {2, 3}, {0, 7}, {2, 3}, {0, 3}, {2, 3}, {0, 31},
{2, 3}, {0, 3}, {2, 3}, {0, 7}, {2, 3}, {0, 3}, {2, 3}, {0, 15},
{2, 3}, {0, 3}, {2, 3}, {0, 7}, {2, 3}, {0, 3}, {2, 3}, {0, 127}, ...]

E.4.3 $te(1) - t$ *for* $q = 3$

The following is $te(1) - t$ for $q = 3$.

The first 60 terms of the simple CF expansion:

$te(1) - t =$ [0, {2:0, 1:2}, {2:3, 1:5, 1:11, 2:13}, {1:0, 2:2},
{1:6, 2:12, 2:14, 1:20, 2:32, 1:38, 1:40, 2:46},

{2:0, 1:2}, {1:3, 2:5, 2:11, 1:13}, {1:0, 2:2},

{2:15, 1:33, 1:39, 1:41, 2:57, 2:59, 2:65, 1:83, 1:95, 2:113, 2:119, 2:121, 1:137, 1:139, 1:145, 2:163},

{2:0, 1:2}, {2:3, 1:5, 1:11, 2:13}, {1:0, 2:2},

{2:6, 1:12, 1:14, 2:20, 1:32, 2:38, 2:40, 1:46},

{2:0, 1:2}, {1:3, 2:5, 2:11, 1:13}, {1:0, 2:2},

{1:42, 2:96, 2:114, 2:120, 2:122, 1:168, 1:174, 1:176, 1:192, 1:194, 1:200, 2:246, 2:248, 2:254, 2:272, 2:284, 1:326, 1:338, 1:356, 1:362, 1:364, 2:410, 2:416, 2:418, 2:434, 2:436, 2:442, 1:488, 1:490, 1:496, 1:514, 2:568},

{2:0, 1:2}, {2:3, 1:5, 1:11, 2:13}, {1:0, 2:2},

{1:6, 2:12, 2:14, 1:20, 2:32, 1:38, 1:40, 2:46},

{2:0, 1:2}, {1:3, 2:5, 2:11, 1:13}, {1:0, 2:2},

{1:15, 2:33, 2:39, 2:41, 1:57, 1:59, 1:65, 2:83, 2:95, 1:113, 1:119, 1:121, 2:137, 2:139, 2:145, 1:163},

{2:0, 1:2}, {2:3, 1:5, 1:11, 2:13}, {1:0, 2:2},

{2:6, 1:12, 1:14, 2:20, 1:32, 2:38, 2:40, 1:46},

{2:0, 1:2}, {1:3, 2:5, 2:11, 1:13}, {1:0, 2:2},

{2:123, 1:285, 1:339, 1:357, 1:363, 1:365, 2:501, 2:519, 2:525, 2:527, 2:573, 2:579, 2:581, 2:597, 2:599, 2:605, 1:735, 1:741, 1:743, 1:759, 1:761, 1:767, 1:813, 1:815, 1:821, 1:839, 1:851, 2:975, 2:977, 2:983, 2:1001, 2:1013, 2:1055, 2:1067, 2:1085, 2:1091, 2:1093, 1:1217, 1:1229, 1:1247, 1:1253, 1:1255, 1:1301, 1:1307, 1:1309, 1:1325, 1:1327, 1:1333, 2:1463, 2:1469, 2:1471, 2:1487, 2:1489, 2:1495, 2:1541, 2:1543, 2:1549, 2:1567, 1:1703, 1:1705, 1:1711, 1:1729, 1:1783, 2:1945},

{2:0, 1:2}, {2:3, 1:5, 1:11, 2:13}, {1:0, 2:2},

{1:6, 2:12, 2:14, 1:20, 2:32, 1:38, 1:40, 2:46},

{2:0, 1:2}, {1:3, 2:5, 2:11, 1:13}, {1:0, 2:2},

{2:15, 1:33, 1:39, 1:41, 2:57, 2:59, 2:65, 1:83, 1:95, 2:113, 2:119, 2:121, 1:137, 1:139, 1:145, 2:163},

{2:0, 1:2}, {2:3, 1:5, 1:11, 2:13}, {1:0, 2:2},

{2:6, 1:12, 1:14, 2:20, 1:32, 2:38, 2:40, 1:46},

{2:0, 1:2}, {1:3, 2:5, 2:11, 1:13}, {1:0, 2:2},

{2:42, 1:96, 1:114, 1:120, 1:122, 2:168, 2:174, 2:176, 2:192, 2:194, 2:200, 1:246, 1:248, 1:254, 1:272, 1:284, 2:326, 2:338, 2:356, 2:362, 2:364, 1:410, 1:416, 1:418, 1:434, 1:436, 1:442, 2:488, 2:490, 2:496, 2:514, 1:568},

{2:0, 1:2}, {2:3, 1:5, 1:11, 2:13}, {1:0, 2:2},

{1:6, 2:12, 2:14, 1:20, 2:32, 1:38, 1:40, 2:46},

{2:0, 1:2}, {1:3, 2:5, 2:11, 1:13}, {1:0, 2:2},

{1:15, 2:33, 2:39, 2:41, 1:57, 1:59, 1:65, 2:83, 2:95, 1:113, 1:119, 1:121, 2:137, 2:139, 2:145, 1:163},

{2:0, 1:2}, {2:3, 1:5, 1:11, 2:13}, {1:0, 2:2}, ...]

E.4.4 $\frac{1}{t}e(1) - \frac{1}{t}$ *for* $q = 3$

The following is $\frac{1}{t}e(1) - \frac{1}{t}$ for $q = 3$.

The first 60 terms of the simple CF expansion:

$\frac{1}{t}e(1) - \frac{1}{t}$ = [0, {2:2, 1:4}, {2:1, 1:3, 1:9, 2:11}, {1:2, 2:4},

{1:4, 2:10, 2:12, 1:18, 2:30, 1:36, 1:38, 2:44},

{2:2, 1:4}, {1:1, 2:3, 2:9, 1:11}, {1:2, 2:4},

{2:13, 1:31, 1:37, 1:39, 2:55, 2:57, 2:63, 1:81, 1:93, 2:111, 2:117, 2:119, 1:135, 1:137, 1:143, 2:161},

{2:2, 1:4}, {2:1, 1:3, 1:9, 2:11}, {1:2, 2:4},

{2:4, 1:10, 1:12, 2:18, 1:30, 2:36, 2:38, 1:44},

{2:2, 1:4}, {1:1, 2:3, 2:9, 1:11}, {1:2, 2:4},

{1:40, 2:94, 2:112, 2:118, 2:120, 1:166, 1:172, 1:174, 1:190, 1:192, 1:198, 2:244, 2:246, 2:252, 2:270, 2:282, 1:324, 1:336, 1:354, 1:360, 1:362, 2:408, 2:414, 2:416, 2:432, 2:434, 2:440, 1:486, 1:488, 1:494, 1:512, 2:566},

{2:2, 1:4}, {2:1, 1:3, 1:9, 2:11}, {1:2, 2:4},

{1:4, 2:10, 2:12, 1:18, 2:30, 1:36, 1:38, 2:44},

{2:2, 1:4}, {1:1, 2:3, 2:9, 1:11}, {1:2, 2:4},

{1:13, 2:31, 2:37, 2:39, 1:55, 1:57, 1:63, 2:81, 2:93, 1:111, 1:117, 1:119, 2:135, 2:137, 2:143, 1:161},

{2:2, 1:4}, {2:1, 1:3, 1:9, 2:11}, {1:2, 2:4},

{2:4, 1:10, 1:12, 2:18, 1:30, 2:36, 2:38, 1:44},

{2:2, 1:4}, {1:1, 2:3, 2:9, 1:11}, {1:2, 2:4},

{2:121, 1:283, 1:337, 1:355, 1:361, 1:363, 2:499, 2:517, 2:523, 2:525, 2:571, 2:577, 2:579, 2:595, 2:597, 2:603, 1:733, 1:739, 1:741, 1:757, 1:759, 1:765, 1:811, 1:813, 1:819, 1:837, 1:849, 2:973, 2:975, 2:981, 2:999, 2:1011, 2:1053, 2:1065, 2:1083, 2:1089, 2:1091, 1:1215, 1:1227, 1:1245, 1:1251, 1:1253, 1:1299, 1:1305, 1:1307, 1:1323, 1:1325, 1:1331, 2:1461, 2:1467, 2:1469, 2:1485, 2:1487, 2:1493, 2:1539, 2:1541, 2:1547, 2:1565, 1:1701, 1:1703, 1:1709, 1:1727, 1:1781, 2:1943},

{2:2, 1:4}, {2:1, 1:3, 1:9, 2:11}, {1:2, 2:4},

{1:4, 2:10, 2:12, 1:18, 2:30, 1:36, 1:38, 2:44},

{2:2, 1:4}, {1:1, 2:3, 2:9, 1:11}, {1:2, 2:4},

{2:13, 1:31, 1:37, 1:39, 2:55, 2:57, 2:63, 1:81, 1:93, 2:111, 2:117, 2:119, 1:135, 1:137, 1:143, 2:161},

{2:2, 1:4}, {2:1, 1:3, 1:9, 2:11}, {1:2, 2:4},

{2:4, 1:10, 1:12, 2:18, 1:30, 2:36, 2:38, 1:44},

{2:2, 1:4}, {1:1, 2:3, 2:9, 1:11}, {1:2, 2:4},

{2:40, 1:94, 1:112, 1:118, 1:120, 2:166, 2:172, 2:174, 2:190, 2:192, 2:198, 1:244, 1:246, 1:252, 1:270, 1:282, 2:324, 2:336, 2:354, 2:360, 2:362, 1:408, 1:414, 1:416, 1:432, 1:434, 1:440, 2:486, 2:488, 2:494, 2:512, 1:566},

{2:2, 1:4}, {2:1, 1:3, 1:9, 2:11}, {1:2, 2:4},

{1:4, 2:10, 2:12, 1:18, 2:30, 1:36, 1:38, 2:44},

{2:2, 1:4}, {1:1, 2:3, 2:9, 1:11}, {1:2, 2:4},

{1:13, 2:31, 2:37, 2:39, 1:55, 1:57, 1:63, 2:81, 2:93, 1:111, 1:117, 1:119, 2:135, 2:137, 2:143, 1:161},

{2:2, 1:4}, {2:1, 1:3, 1:9, 2:11}, {1:2, 2:4, ...]

E.5 Fancy Hurwitz Example (FHE3-1), $q = 3$

The groundwork of explaining the theoretical and computational process of evaluating simple Hurwitz numbers is laid. To get fancy in a spirit of fun, look at an interesting, simple Hurwitz number for $q = 3$.

This first fancy Hurwitz example will be discussed more in Chapter 3.2.4.5, p. 30, with the proof of Theorem 2.

The choices of x, y, z, w and f were made to be of low degree, irreducible and prime.

Let $x = t + 2$, $y = 2t$, $z =$ t, $w = t + 1$, $\theta = 1$, $f = t$. Then:

$$(\tfrac{x}{y})e(\tfrac{1}{f}) + (\tfrac{z}{w}) = (\tfrac{t+2}{2t})e(\tfrac{1}{t}) + \tfrac{t}{t+1}$$

The first 60 terms of the simple CF expansion:

$(\frac{t+2}{2t})e(\frac{1}{t}) + \frac{t}{t+1}$ = [1, {1:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {1:0, 2:1}, {1:0, 2:1},

{2:4, 2:5, 2:6, 2:7, 2:8, 2:9, 1:12, 1:13, 1:14, 1:15, 1:16, 1:17},

{2:0, 1:1}, {2:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {2:0, 2:1},

{1:13, 2:14, 2:19, 1:20, 2:21, 1:22, 1:27, 2:28, 2:39, 1:40, 1:45, 2:46, 1:47, 2:48, 2:53, 1:54},

{1:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {1:0, 2:1}, {1:0, 2:1},

{1:4, 1:5, 1:6, 1:7, 1:8, 1:9, 2:12, 2:13, 2:14, 2:15, 2:16, 2:17},

{2:0, 1:1}, {2:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {2:0, 2:1},

{2:40, 1:41, 1:58, 2:59, 1:64, 2:65, 1:66, 2:67, 2:82, 1:83, 2:84, 1:85, 2:90, 1:91, 1:108, 2:109, 1:120, 2:121, 2:138, 1:139, 2:144, 1:145, 2:146, 1:147, 1:162, 2:163, 1:164, 2:165, 1:170, 2:171, 2:188, 1:189},

{1:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {1:0, 2:1}, {1:0, 2:1},

{2:4, 2:5, 2:6, 2:7, 2:8, 2:9, 1:12, 1:13, 1:14, 1:15, 1:16, 1:17},

{2:0, 1:1}, {2:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {2:0, 2:1},

{2:13, 1:14, 1:19, 2:20, 1:21, 2:22, 2:27, 1:28, 1:39, 2:40, 2:45, 1:46, 2:47, 1:48, 1:53, 2:54},

{1:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {1:0, 2:1}, {1:0, 2:1},

{1:4, 1:5, 1:6, 1:7, 1:8, 1:9, 2:12, 2:13, 2:14, 2:15, 2:16, 2:17},

{2:0, 1:1}, {2:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {2:0, 2:1},

{1:121, 2:122, 2:175, 1:176, 2:193, 1:194, 2:199, 1:200, 2:201, 1:202, 1:247, 2:248, 1:253, 2:254, 1:255, 2:256, 1:271, 2:272, 1:273, 2:274, 1:279, 2:280, 2:325, 1:326, 2:327, 1:328, 2:333, 1:334, 2:351, 1:352, 2:363, 1:364, 1:405, 2:406, 1:417, 2:418, 1:435, 2:436, 1:441, 2:442, 1:443, 2:444, 2:489, 1:490, 2:495, 1:496, 2:497, 1:498, 2:513, 1:514, 2:515, 1:516, 2:521, 1:522, 1:567, 2:568, 1:569, 2:570, 1:575, 2:576, 1:593, 2:594, 2:647, 1:648},

{1:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {1:0, 2:1}, {1:0, 2:1},

{2:4, 2:5, 2:6, 2:7, 2:8, 2:9, 1:12, 1:13, 1:14, 1:15, 1:16, 1:17},

{2:0, 1:1}, {2:0, 1:1}, {1:0, 2:1}, {2:0, 1:1}, {2:0, 2:1}, ...]

E.6 Fancy Hurwitz Example (FHE3-2), $q = 3$

Here is a second "fancy" example for $q = 3$.

Let $x = t^2$, $y = t^2{+}t{+}1$, $z = t + 1$, $w = t$, $\theta = 1$, $f = t$. Then:

$$(\frac{x}{y})e(\frac{1}{f}) + (\frac{z}{w}) = (\frac{t^2}{t^2+t+1})e(\frac{1}{t}) + \frac{t+1}{t}$$

The first 60 terms of the simple CF expansion:

$(\frac{t^2}{t^2+t+1})e(\frac{1}{t}) + \frac{t+1}{t}$ = [1, {1:0, 2:1}, {2:1}, {1:0, 1:1}, {1:0, 1:1, 1:2, 1:3},

{1:7, 2:8, 2:9, 2:10, 2:11, 2:12, 2:13, 2:14, 1:15},

{2:0, 2:1, 2:2, 2:3}, {2:0, 2:1}, {1:1}, {2:0, 1:1},

{1:16, 2:17, 1:19, 2:20, 2:24, 1:25, 2:27, 1:28, 2:42, 1:43, 2:45, 1:46, 1:50, 2:51, 1:53, 2:54},

{1:0, 2:1}, {2:1}, {1:0, 1:1}, {1:0, 1:1, 1:2, 1:3},

{2:7, 1:8, 1:9, 1:10, 1:11, 1:12, 1:13, 1:14, 2:15},

{2:0, 2:1, 2:2, 2:3}, {2:0, 2:1}, {1:1}, {2:0, 1:1},

{2:43, 1:44, 2:46, 1:47, 2:49, 1:50, 2:52, 1:53, 2:55, 1:56, 2:58, 1:59, 1:67, 2:68, 1:69, 2:71, 1:72, 2:74, 1:75, 2:77, 1:78, 2:80, 1:81, 2:83, 1:84, 2:85, 2:93, 1:94, 2:96, 1:97, 2:99, 1:100, 2:102, 1:103, 2:105, 1:106, 2:108, 1:109, 1:123, 2:124, 1:126, 2:127, 1:129, 2:130, 1:132, 2:133, 1:135, 2:136, 1:138, 2:139, 2:147, 1:148, 2:149, 1:151, 2:152, 1:154, 2:155, 1:157, 2:158, 1:160, 2:161, 1:163, 2:164, 1:165, 1:173, 2:174, 1:176, 2:177, 1:179, 2:180, 1:182, 2:183, 1:185, 2:186, 1:188, 2:189},

{1:0, 2:1}, {2:1}, {1:0, 1:1}, {1:0, 1:1, 1:2, 1:3},

{1:7, 2:8, 2:9, 2:10, 2:11, 2:12, 2:13, 2:14, 1:15},

{2:0, 2:1, 2:2, 2:3}, {2:0, 2:1}, {1:1}, {2:0, 1:1},

{2:16, 1:17, 2:19, 1:20, 1:24, 2:25, 1:27, 2:28, 1:42, 2:43, 1:45, 2:46, 2:50, 1:51, 2:53, 1:54},

{1:0, 2:1}, {2:1}, {1:0, 1:1}, {1:0, 1:1, 1:2, 1:3},

{2:7, 1:8, 1:9, 1:10, 1:11, 1:12, 1:13, 1:14, 2:15},

{2:0, 2:1, 2:2, 2:3}, {2:0, 2:1}, {1:1}, {2:0, 1:1},

{1:124, 2:125, 1:127, 2:128, 1:130, 2:131, 1:133, 2:134, 1:136, 2:137, 1:139, 2:140, 1:142, 2:143, 1:145, 2:146, 1:148, 2:149, 1:151, 2:152, 1:154, 2:155, 1:157, 2:158, 1:160, 2:161, 1:163, 2:164, 1:166, 2:167, 1:169, 2:170, 1:172, 2:173, 1:175, 2:176, 2:196, 1:197, 2:199, 1:200, 1:202, 2:203, 2:204, 2:205, 2:206, 2:207, 2:208, 2:209, 2:210, 2:211, 2:212, 2:213, 2:214, 2:215, 2:216, 2:217, 2:218, 2:219, 2:220, 2:221, 2:222, 2:223, 2:224, 2:225, 2:226, 2:227, 2:228, 2:229, 2:230, 2:231, 2:232, 2:233, 2:234, 2:235, 2:236, 2:237, 2:238, 2:239, 2:240, 2:241, 2:242, 2:243, 2:244, 2:245, 2:246, 2:247, 2:248, 2:249, 1:251, 2:252, 1:254, 2:255, 1:256, 1:274, 2:275, 1:276, 2:278, 1:279, 2:281, 2:282, 2:283, 2:284, 2:285, 2:286, 2:287, 2:288, 2:289, 2:290, 2:291, 2:292, 2:293, 2:294, 2:295, 2:296, 2:297, 2:298, 2:299, 2:300, 2:301, 2:302, 2:303, 2:304, 2:305, 2:306, 2:307, 2:308, 2:309, 2:310, 2:311, 2:312, 2:313, 2:314, 2:315, 2:316, 2:317, 2:318, 2:319, 2:320, 2:321, 2:322, 2:323, 2:324, 2:325, 2:326, 2:327, 1:328, 1:330, 2:331, 1:333, 2:334, 2:354, 1:355, 2:357, 1:358, 2:360, 1:361, 2:363, 1:364, 1:366, 2:367, 1:369, 2:370, 1:372, 2:373, 1:375, 2:376, 1:378, 2:379, 1:381, 2:382, 1:384, 2:385, 1:387, 2:388, 1:390, 2:391, 1:393, 2:394, 1:396, 2:397, 1:399, 2:400, 1:402, 2:403, 1:405, 2:406, 2:408, 1:409, 2:411, 1:412, 2:414, 1:415, 2:417, 1:418, 1:438, 2:439, 1:441, 2:442, 2:444, 1:445, 1:446, 1:447, 1:448, 1:449, 1:450, 1:451, 1:452, 1:453, 1:454, 1:455, 1:456, 1:457, 1:458, 1:459, 1:460, 1:461, 1:462, 1:463, 1:464, 1:465, 1:466, 1:467, 1:468, 1:469, 1:470, 1:471, 1:472, 1:473, 1:474, 1:475, 1:476, 1:477, 1:478, 1:479, 1:480, 1:481, 1:482, 1:483, 1:484, 1:485, 1:486, 1:487, 1:488, 1:489, 1:490, 1:491, 2:493, 1:494, 2:496, 1:497, 2:498, 2:516, 1:517, 2:518, 1:520, 2:521, 1:523, 1:524, 1:525, 1:526, 1:527, 1:528, 1:529, 1:530, 1:531, 1:532, 1:533, 1:534, 1:535, 1:536, 1:537, 1:538, 1:539, 1:540, 1:541, 1:542, 1:543, 1:544, 1:545, 1:546, 1:547, 1:548, 1:549, 1:550, 1:551, 1:552, 1:553, 1:554, 1:555, 1:556, 1:557, 1:558, 1:559, 1:560, 1:561, 1:562, 1:563, 1:564, 1:565, 1:566, 1:567, 1:568, 1:569, 2:570, 2:572, 1:573, 2:575, 1:576, 1:596, 2:597, 1:599, 2:600, 1:602, 2:603, 1:605, 2:606, 1:608, 2:609, 1:611, 2:612, 1:614, 2:615, 1:617, 2:618, 1:620, 2:621, 1:623, 2:624, 1:626, 2:627, 1:629, 2:630, 1:632, 2:633, 1:635, 2:636, 1:638, 2:639, 1:641, 2:642, 1:644, 2:645, 1:647, 2:648},

{1:0, 2:1}, {2:1}, {1:0, 1:1}, {1:0, 1:1, 1:2, 1:3},

{1:7, 2:8, 2:9, 2:10, 2:11, 2:12, 2:13, 2:14, 1:15},

{2:0, 2:1, 2:2, 2:3}, {2:0, 2:1}, {1:1}, {2:0, 1:1},

{1:16, 2:17, 1:19, 2:20, 2:24, 1:25, 2:27, 1:28, 2:42, 1:43, 2:45, 1:46, 1:50, 2:51, 1:53, 2:54},

{1:0, 2:1}, {2:1}, {1:0, 1:1}, {1:0, 1:1, 1:2, 1:3},

{2:7, 1:8, 1:9, 1:10, 1:11, 1:12, 1:13, 1:14, 2:15},

{2:0, 2:1, 2:2, 2:3}, {2:0, 2:1}, {1:1}, {2:0, 1:1, ...]

E.7 Fancy Hurwitz Example (FHE2-1), $q = 2$

Now, examine an interesting simple Hurwitz number for $q = 2$.

The choices of x, y, z, w, θ and f were made to be of low degree, irreducible and prime.

Let $x = t^2$, $y = t^2 + 1$, $z = 0$, $w = 1$, $\theta = t + 1$, $f = t$. Then:

$$(\tfrac{x}{y})e(\tfrac{\theta}{f}) + (\tfrac{z}{w}) = (\tfrac{t^2}{t^2+1})e(\tfrac{t+1}{t})$$

Note that the discriminant $xw - yz = t^2 \neq 0$.

The first 193 (= 3*64 + 1) terms of the simple CF expansion:

$(\frac{t^2}{t^2+1})e(\frac{t+1}{t})$ = [1, {1}, {3, 6}, {1}, {0, 1, 2, 4, 6, 8}, {1, 2}, {0, 1, 2, 4}, {1}, {3, 18}, {1}, {0, 1, 2, 4}, {1, 2}, {0, 1, 2, 4, 6, 8}, {1}, {3, 6}, {1},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32},

{1, 2}, {0, 1, 2, 4}, {1}, {3, 10}, {1}, {0, 1, 2, 4}, {1, 2},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16},

{1}, {3, 6}, {1}, {0, 1, 2, 4, 6, 8}, {1, 2}, {0, 1, 2, 4}, {1}, {3, 66}, {1}, {0, 1, 2, 4}, {1, 2}, {0, 1, 2, 4, 6, 8}, {1}, {3, 6}, {1},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16},

{1, 2}, {0, 1, 2, 4}, {1}, {3, 10}, {1}, {0, 1, 2, 4}, {1, 2},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32},

{1}, {3, 6}, {1}, {0, 1, 2, 4, 6, 8}, {1, 2}, {0, 1, 2, 4}, {1}, {3, 18}, {1}, {0, 1, 2, 4}, {1, 2}, {0, 1, 2, 4, 6, 8}, {1}, {3, 6}, {1},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120, 122, 124, 126, 128},

{1, 2}, {0, 1, 2, 4}, {1}, {3, 10}, {1}, {0, 1, 2, 4}, {1, 2},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16},

{1}, {3, 6}, {1}, {0, 1, 2, 4, 6, 8}, {1, 2}, {0, 1, 2, 4}, {1}, {3, 34}, {1}, {0, 1, 2, 4}, {1, 2}, {0, 1, 2, 4, 6, 8}, {1}, {3, 6}, {1},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16},

{1, 2}, {0, 1, 2, 4}, {1}, {3, 10}, {1}, {0, 1, 2, 4}, {1, 2},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64},

{1}, {3, 6}, {1}, {0, 1, 2, 4, 6, 8}, {1, 2}, {0, 1, 2, 4}, {1}, {3, 18}, {1}, {0, 1, 2, 4}, {1, 2}, {0, 1, 2, 4, 6, 8}, {1}, {3, 6}, {1},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32},

{1, 2}, {0, 1, 2, 4}, {1}, {3, 10}, {1}, {0, 1, 2, 4}, {1, 2},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16},

{1}, {3, 6}, {1}, {0, 1, 2, 4, 6, 8}, {1, 2}, {0, 1, 2, 4}, {1}, {3, 258}, {1}, {0, 1, 2, 4}, {1, 2}, {0, 1, 2, 4, 6, 8}, {1}, {3, 6}, {1},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16},

{1, 2}, {0, 1, 2, 4}, {1}, {3, 10}, {1}, {0, 1, 2, 4}, {1, 2},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32},

{1}, {3, 6}, {1}, {0, 1, 2, 4, 6, 8}, {1, 2}, {0, 1, 2, 4}, {1}, {3, 18}, {1}, {0, 1, 2, 4}, {1, 2}, {0, 1, 2, 4, 6, 8}, {1}, {3, 6}, {1},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64},

{1, 2}, {0, 1, 2, 4}, {1}, {3, 10}, {1}, {0, 1, 2, 4}, {1, 2},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16},

{1}, {3, 6}, {1}, {0, 1, 2, 4, 6, 8}, {1, 2}, {0, 1, 2, 4}, {1}, {3, 34}, {1}, {0, 1, 2, 4}, {1, 2}, {0, 1, 2, 4, 6, 8}, {1}, {3, 6}, {1},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16},

{1, 2}, {0, 1, 2, 4}, {1}, {3, 10}, {1}, {0, 1, 2, 4}, {1, 2},

{0, 1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120, 122, 124, 126, 128}, ...]

E.8 Fancy Hurwitz Example (FHE2-2), $q = 2$

The following is a second interesting simple Hurwitz number for $q = 2$.

Let $x = t^2$, $y = t^2 + t + 1$, $z = 0$, $w = 1$, $\theta = t + 1$, $f = t$. Then:

$$(\tfrac{x}{y})e(\tfrac{\theta}{f}) + (\tfrac{z}{w}) = (\tfrac{t^2}{t^2+t+1})e(\tfrac{t+1}{t})$$

Note that the discriminant $xw - yz = t^2 \neq 0$.

The first 193 (= 3*64 + 1) terms of the simple CF expansion:

$(\tfrac{t^2}{t^2+t+1})e(\tfrac{t+1}{t})$ = [1, {1, 2, 3}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1}, {0, 4, 5, 6}, {1}, {3, 4, 5, 6, 7, 8}, {1},

{0, 1, 2, 6, 7, 8, 12, 13, 14},

{1, 2}, {0, 1, 2, 5, 6}, {1}, {0, 2, 4, 5, 7, 8}, {1, 2, 3}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1},

{0, 4, 5, 6, 10, 11, 12, 16, 17, 18, 22, 23, 24, 28, 29, 30},

{1}, {3, 4, 5, 6, 7, 8}, {1}, {0, 4, 5, 6}, {1}, {1}, {0, 1}, {0, 1}, {0, 1, 2}, {1, 2, 3},

{3, 4, 6, 7, 9, 10, 12, 13, 15, 16},

{1, 2, 3}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1}, {0, 4, 5, 6}, {1}, {3, 4, 5, 6, 7, 8}, {1},

{0, 1, 2, 6, 7, 8, 12, 13, 14, 18, 19, 20, 24, 25, 26, 30, 31, 32, 36, 37, 38, 42, 43, 44, 48, 49, 50, 54, 55, 56, 60, 61, 62}, {1, 2}, {0, 1, 2, 5, 6}, {1}, {0, 2, 4, 5, 7, 8}, {1, 2, 3}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1},

{0, 1, 2, 6, 7, 8, 12, 13, 14},

{0, 2}, {0, 1}, {3, 4}, {0, 1}, {0, 2}, {0, 4, 5, 6}, {1, 2}, {0, 1, 2, 5, 6}, {1},

{0, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26, 28, 29, 31, 32},

{1, 2, 3}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1}, {0, 4, 5, 6}, {1}, {3, 4, 5, 6, 7, 8}, {1},

{0, 1, 2, 6, 7, 8, 12, 13, 14},

{1, 2}, {0, 1, 2, 5, 6}, {1}, {0, 2, 4, 5, 7, 8}, {1, 2, 3}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1},

{0, 4, 5, 6, 10, 11, 12, 16, 17, 18, 22, 23, 24, 28, 29, 30, 34, 35, 36, 40, 41, 42, 46, 47, 48, 52, 53, 54, 58, 59, 60, 64, 65, 66, 70, 71, 72, 76, 77, 78, 82, 83, 84, 88, 89, 90, 94, 95, 96, 100, 101, 102, 106, 107, 108, 112, 113, 114, 118, 119, 120, 124, 125, 126},

{1}, {3, 4, 5, 6, 7, 8}, {1}, {0, 4, 5, 6}, {1}, {1}, {0, 1}, {0, 1}, {0, 1, 2}, {1, 2, 3},

{3, 4, 6, 7, 9, 10, 12, 13, 15, 16},

{1, 2, 3}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1}, {0, 4, 5, 6}, {1}, {3, 4, 5, 6, 7, 8}, {1},

{0, 4, 5, 6, 10, 11, 12, 16, 17, 18, 22, 23, 24, 28, 29, 30},

{1}, {1}, {0, 1}, {0, 1}, {0, 1, 2}, {1, 2, 3}, {0, 2, 4, 5, 7, 8}, {1}, {0, 1, 2, 5, 6}, {1, 2},

{0, 1, 2, 6, 7, 8, 12, 13, 14},

{1}, {3, 4, 5, 6, 7, 8}, {1}, {0, 4, 5, 6}, {1}, {1}, {0, 1}, {0, 1}, {0, 1, 2}, {1, 2, 3},

{3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 21, 22, 24, 25, 27, 28, 30, 31, 33, 34, 36, 37, 39, 40, 42, 43, 45, 46, 48, 49, 51, 52, 54, 55, 57, 58, 60, 61, 63, 64},

{1, 2, 3}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1}, {0, 4, 5, 6}, {1}, {3, 4, 5, 6, 7, 8}, {1},

{0, 1, 2, 6, 7, 8, 12, 13, 14},

{1, 2}, {0, 1, 2, 5, 6}, {1}, {0, 2, 4, 5, 7, 8}, {1, 2, 3}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1},

{0, 4, 5, 6, 10, 11, 12, 16, 17, 18, 22, 23, 24, 28, 29, 30},

{1}, {3, 4, 5, 6, 7, 8}, {1}, {0, 4, 5, 6}, {1}, {1}, {0, 1}, {0, 1}, {0, 1, 2}, {1, 2, 3},

{3, 4, 6, 7, 9, 10, 12, 13, 15, 16},

{1, 2, 3}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1}, {0, 4, 5, 6}, {1}, {3, 4, 5, 6, 7, 8}, {1},

{0, 1, 2, 6, 7, 8, 12, 13, 14, 18, 19, 20, 24, 25, 26, 30, 31, 32, 36, 37, 38, 42, 43, 44, 48, 49, 50, 54, 55, 56, 60, 61, 62, 66, 67, 68, 72, 73, 74, 78, 79, 80, 84, 85, 86, 90, 91, 92, 96, 97, 98, 102, 103, 104, 108, 109, 110, 114, 115, 116, 120, 121, 122, 126, 127, 128, 132, 133, 134, 138, 139, 140, 144, 145, 146, 150, 151, 152, 156, 157, 158, 162, 163, 164, 168, 169, 170, 174, 175, 176, 180, 181, 182, 186, 187, 188, 192, 193, 194, 198, 199, 200, 204, 205, 206, 210, 211, 212, 216, 217, 218, 222, 223, 224, 228, 229, 230, 234, 235, 236, 240, 241, 242, 246, 247, 248, 252, 253, 254},

{1, 2}, {0, 1, 2, 5, 6}, {1}, {0, 2, 4, 5, 7, 8}, {1, 2, 3}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1},

{0, 1, 2, 6, 7, 8, 12, 13, 14},

{0, 2}, {0, 1}, {3, 4}, {0, 1}, {0, 2}, {0, 4, 5, 6}, ...]

E.9 General Hurwitz Number Examples, $q = 2$

The first part of this "Hurwitz Numbers and Development" Appendix was devoted to simple Hurwitz numbers. These are the ones which Dr. Thakur feels give the most "pure," interesting patterns.

Now, general Hurwitz expansions will be discussed. General Hurwitz numbers can be converted to the simple Hurwitz form by Mobïus transformations which keeps the tails the same. Dr. Thakur proved this in Chapter 3.2.4.7, p. 34.

General Hurwitz numbers are of the form:

$$\frac{xe(\frac{\theta}{f})+y}{ze(\frac{\theta}{f})+w}$$

The process for evaluating them parallels the process of evaluating simple Hurwitz expansions. This is seen above, in this appendix, at E.2, p. 80.

Evaluate:

$$\frac{x(\frac{(\frac{\theta}{f})}{1}+\sum_{i=1}^{7}\frac{(\frac{\theta}{f})^{q^i}}{D_i})+y}{z(\frac{(\frac{\theta}{f})}{1}+\sum_{i=1}^{7}\frac{(\frac{\theta}{f})^{q^i}}{D_i})+w}$$

Reduce this new expression by modulo q if appropriate. Then, convert the rational function form of the existing expression to its simple CF form with the "Rational Function to Simple CF Expansion Algorithm" outlined in Item 3 of the process.

The following are a couple of interesting examples of general Hurwitz number expansions for $q = 2$.

E.9.1 $\frac{t^2e(t)+1}{(t+1)e(t)+t^3}$ *for q = 2*

Let $x = t^2$, $y = 1$, $z = t + 1$, $w = t^3$, $\theta = t$, $f = 1$. Then:

$$\frac{xe(\frac{\theta}{f})+y}{ze(\frac{\theta}{f})+w} = \frac{t^2e(t)+1}{(t+1)e(t)+t^3}$$

Note that the discriminant $xw - yz = t^5 + t + 1 \neq 0$.

The first 193 (= 3*64 + 1) terms of the general CF expansion:

$\frac{t^2e(t)+1}{(t+1)e(t)+t^3}$ = [1, {0, 1, 2}, {0, 1}, {0, 1}, {0, 1}, {0, 2, 3}, {1}, {1, 4, 5}, {0, 1, 3, 4}, {2}, {1, 4, 5, 9}, {0, 1}, {2, 4}, {0, 1}, {0, 1}, {0, 1, 2}, {1}, {3}, {2}, {2, 4}, {0, 1}, {0, 1}, {2},

{0, 4, 9, 10, 11, 12, 13, 15, 17, 20, 21, 25},

{0, 1}, {1}, {0, 1, 2}, {2}, {1, 3}, {1}, {1}, {1}, {1}, {1, 2}, {0, 1}, {1}, {3}, {1}, {1}, {1, 3, 6, 7, 11}, {2}, {2, 4}, {0, 1}, {0, 1}, {2}, {1}, {1, 2}, {0, 2}, {1, 2}, {0, 1}, {0, 1}, {1, 2}, {0, 1},

{0, 1, 2, 3, 5, 7, 10, 11, 15, 20, 21, 22, 23, 24, 26, 28, 31, 32, 36, 41, 42, 43, 44, 45, 47, 49, 52, 53, 57},

{0, 1, 2}, {1}, {1, 2, 3}, {0, 1}, {0, 2}, {0, 1}, {3}, {1, 2, 3}, {0, 1}, {0, 2}, {0, 1}, {1}, {1}, {0, 1}, {1, 4, 5, 9}, {0, 1, 2}, {0, 1, 2}, {1}, {0, 1, 2}, {0, 3, 4}, {1}, {1, 3}, {1}, {1}, {2}, {0, 1}, {1, 2},

{1, 2, 6, 11, 12, 13, 14, 15, 17, 19, 22, 23, 27},

{1, 2, 3}, {0, 1}, {0, 2}, {0, 1}, {1}, {1}, {0, 1}, {1}, {0, 1}, {1, 2}, {0, 1}, {0, 1}, {1, 2}, {0, 2}, {1, 2}, {1, 4, 5, 9}, {0, 1}, {0, 1}, {0, 1, 3, 4}, {1}, {2, 3}, {3}, {0, 1}, {1, 2}, {0, 1}, {0, 1, 2}, {0, 1}, {1}, {1, 2},

{0, 1, 2, 3, 4, 6, 8, 11, 12, 16, 21, 22, 23, 24, 25, 27, 29, 32, 33, 37, 42, 43, 44, 45, 46, 48, 50, 53, 54, 58, 63, 64, 65, 66, 67, 69, 71, 74, 75, 79, 84, 85, 86, 87, 88, 90, 92, 95, 96, 100, 105, 106, 107, 108, 109, 111, 113, 116, 117, 121},

{1, 2}, {0, 2}, {1, 2}, {0, 1}, {0, 1}, {1, 2}, {0, 1}, {1}, {0, 1}, {1}, {1}, {0, 1}, {0, 2}, {0, 1}, {1, 2, 3}, {1, 3, 6, 7, 11}, {0, 1}, {1}, {0, 1, 2}, {0, 1}, {0, 1}, {0, 1}, {0, 2, 3}, {1}, {1, 4, 5}, {0, 1, 3, 4}, {2},

{0, 4, 9, 10, 11, 12, 13, 15, 17, 20, 21, 25},

{0, 1}, {1}, {1}, {0, 1}, {0, 2}, {0, 1}, {1, 2, 3}, {3}, {0, 1}, {0, 2}, {0, 1}, {1, 2, 3}, {1}, {0, 1, 2}, {1, 4, 5, 9}, {0, 1}, {1}, {0, 1, 2}, {2}, {1, 3}, {1}, {1}, {1}, {1}, {1, 2}, {0, 1}, {1}, {3}, {1}, {1},

{1, 2, 3, 4, 5, 7, 9, 12, 13, 17, 22, 23, 24, 25, 26, 28, 30, 33, 34, 38, 43, 44, 45, 46, 47, 49, 51, 54, 55, 59},

{1}, {1}, {3}, {1}, {0, 1}, {1, 2}, {1}, {1}, {1}, {1}, {1, 3}, {2}, {0, 1, 2}, {1}, {0, 1}, {1, 4, 5, 9}, {0, 1, 2}, {1}, {1, 2, 3}, {0, 1}, {0, 2}, {0, 1}, ...]

E.9.2 $\frac{te(t+1)+t^2}{(t^2+1)e(t+1)+t+1}$ *for* $q = 2$

Let $x = t$, $y = t^2$, $z = t^2 + 1$, $w = t + 1$, $\theta = t + 1$, $f = 1$. Then:

$$\frac{xe(\frac{\theta}{f})+y}{ze(\frac{\theta}{f})+w} = \frac{te(t+1)+t^2}{(t^2+1)e(t+1)+t+1}$$

Note that the discriminant $xw - yz = t^4 + t \neq 0$.

The first 193 (= 3*64 + 1) terms of the general CF expansion:

$\frac{te(t+1)+t^2}{(t^2+1)e(t+1)+t+1}$ = [0, {1, 3}, {0, 1}, {0, 1, 2}, {0, 1}, {1}, {0, 2}, {0, 2}, {1, 2}, {1}, {1}, {1, 2}, {0, 2}, {3, 5, 6, 7, 8, 10}, {0, 3, 5, 6}, {1, 2}, {0, 1}, {1, 4}, {1}, {3}, {0, 1}, {0, 1, 2}, {0, 1}, {1},

{0, 2, 7, 9, 10, 11, 12, 14, 19, 21, 22, 23, 24, 26},

{0, 2}, {1, 2}, {1}, {1}, {1, 2}, {0, 2}, {0, 2}, {1}, {0, 1}, {0, 1, 2}, {0, 1}, {3}, {1}, {0, 3, 6, 9, 12}, {1}, {3}, {0, 1}, {0, 1, 2}, {0, 1}, {1}, {0, 2}, {0, 2}, {1, 2}, {1}, {1}, {1, 2}, {0, 2},

{3, 5, 6, 7, 8, 10, 15, 17, 18, 19, 20, 22, 27, 29, 30, 31, 32, 34, 39, 41, 42, 43, 44, 46, 51, 53, 54, 55, 56, 58},

{0, 3, 5, 6}, {1, 2}, {0, 1}, {1, 4}, {1}, {3}, {0, 1}, {0, 1, 2}, {0, 1}, {1}, {3, 5, 6, 7, 8, 10}, {0, 1}, {1, 2}, {2}, {2}, {1, 2}, {0, 1}, {0, 2}, {0, 3, 5, 6}, {1, 2}, {0, 1},

{1, 4, 7, 10, 13, 16, 19, 22, 25, 28},

{1}, {3}, {0, 1}, {0, 1, 2}, {0, 1}, {1}, {0, 2}, {0, 2}, {1, 2}, {1}, {1}, {1, 2}, {0, 2}, {3, 5, 6, 7, 8, 10}, {0, 3, 5, 6}, {1, 2}, {0, 1}, {1, 4}, {1}, {3}, {0, 1}, {0, 1, 2}, {0, 1}, {1},

{0, 2, 7, 9, 10, 11, 12, 14, 19, 21, 22, 23, 24, 26, 31, 33, 34, 35, 36, 38, 43, 45, 46, 47, 48, 50, 55, 57, 58, 59, 60, 62, 67, 69, 70, 71, 72, 74, 79, 81, 82, 83, 84, 86, 91, 93, 94, 95, 96, 98, 103, 105, 106, 107, 108, 110, 115, 117, 118, 119, 120, 122},

{0, 2}, {1, 2}, {1}, {1}, {1, 2}, {0, 2}, {0, 2}, {1}, {0, 1}, {0, 1, 2}, {0, 1}, {3}, {1}, {0, 3, 6, 9, 12}, {1}, {3}, {0, 1}, {0, 1, 2}, {0, 1}, {1}, {0, 2}, {0, 2}, {1, 2}, {1}, {1}, {1, 2}, {0, 2}, {0, 2, 7, 9, 10, 11, 12, 14, 19, 21, 22, 23, 24, 26},

{1}, {0, 1}, {0, 1, 2}, {0, 1}, {3}, {1}, {1, 4}, {0, 1}, {1, 2}, {0, 3, 5, 6}, {3, 5, 6, 7, 8, 10}, {0, 2}, {1, 2}, {1}, {1}, {1, 2}, {0, 2}, {0, 2}, {1}, {0, 1}, {0, 1, 2}, {0, 1}, {3}, {1},

{0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60},

{1}, {3}, {0, 1}, {0, 1, 2}, {0, 1}, {1}, {0, 2}, {0, 2}, {1, 2}, {1}, {1}, {1, 2}, {0, 2}, {3, 5, 6, 7, 8, 10}, {0, 3, 5, 6}, {1, 2}, {0, 1}, {1, 4}, {1}, {3}, {0, 1}, {0, 1, 2}, {0, 1}, {1},

{0, 2, 7, 9, 10, 11, 12, 14, 19, 21, 22, 23, 24, 26},

{0, 2}, {1, 2}, {1}, {1}, {1, 2}, {0, 2}, {0, 2}, {1}, {0, 1}, {0, 1, 2}, {0, 1}, {3}, {1}, {0, 3, 6, 9, 12}, {1}, ...]

Appendix F

ARTICLE 1996 THEOREM 4 GENERAL CASE PROOF WORK-UP

This appendix corroborates the Article 1996 Theorem 4 general case proof for the examples of $n = 3$ and $n = 4$. This proof can be seen in Chapter 3.2.7.5, p. 50.

F.1 General Case Work-up for $n = 3$

First, let $n = 3$.

Claim. $[0, \overrightarrow{X_3}] + [0, \overleftarrow{X_3}] = 1/t^2$.

First of all, note that $\overrightarrow{X_3}$ is defined to be:

$\sum_{i=0}^{1} 1/(d_i t^3) = \frac{1}{t^3} + \frac{1}{d_1 t^3} = \frac{1+t+t^2}{t^4+t^5} = [0,\, 1 + t + t^3,\, 1 + t + t^2] = [0, \overrightarrow{X_3}]$

$[0, \overleftarrow{X_3}] = [0,\, 1 + t + t^2,\, 1 + t + t^3] = \frac{1+t+t^3}{t^4+t^5}$

$[0, \overrightarrow{X_3}] + [0, \overleftarrow{X_3}] = \frac{1+t+t^2}{t^4+t^5} + \frac{1+t+t^3}{t^4+t^5} = \frac{t^2+t^3}{t^4+t^5} = \frac{1}{t^2}$, which corroborates the stated claim.

$q_{k_3} = t^3 d_1 = t^4 + t^5$ and $p_{k_3} = d_1(1 + \frac{1}{d_1}) = d_1 + \frac{d_1}{d_1} = 1 + t + t^2$.

$Q = td_1 + p_{k_3} = 1 + t + t^3$.

$P = \frac{1+p_{k_3} Q}{q_{k_3}} = \frac{1+(1+t+t^2)(1+t+t^3)}{t^4+t^5}$. Now,

$$P = \frac{1+p_{k_3}^2+td_1 p_{k_3}}{t^3 d_1} = \frac{1+(d_1+\frac{d_1}{d_1})^2+td_1(d_1+\frac{d_1}{d_1})}{t^3 d_1} = \frac{d_1+t(d_1+\frac{d_1}{d_1})}{t^3}$$
$$= \frac{t+t^2+t(1+t+t^2)}{t^3} = \frac{t^3}{t^3} = 1$$

It is seen that P is an integer.

$Pq_{k_3} - Qp_{k_3} = 1(t^4 + t^5) - (1 + t + t^3)(1 + t + t^2) = 1$

By continued fraction fact (C), (Chapter 2.2.1, p. 10), $p_{k_3-1} q_{k_3} - q_{k_3-1} p_{k_3} = 1$, so that $p_{k_3}(q_{k_3-1} - Q) = q_{k_3}(p_{k_3-1} - P)$. The degree of $p_{k_3-1} - P$ is less than that of p_{k_3}. Hence, $P = p_{k_3-1}$.

F.2 General Case Work-up for $n = 4$

Now, replicate the above argument and let $n = 4$.

Claim. $[0, \overrightarrow{X_4}] + [0, \overleftarrow{X_4}] = 1/t^3$.

First of all, note that $\overrightarrow{X_4}$ is defined to be:

$\sum_{i=0}^{2} 1/(d_i t^4) = \frac{1}{t^4} + \frac{1}{d_1 t^4} + \frac{1}{d_2 t^4} = \frac{1+t^2+t^8}{t^7+t^9+t^{10}+t^{12}}$

$= [0, t + t^2 + t^4, 1 + t, 1 + t, 1 + t, 1 + t^2, t + t^2 + t^3] = [0, \overrightarrow{X_4}]$

$[0, \overleftarrow{X_4}] = [0, t + t^2 + t^3, 1 + t^2, 1 + t, 1 + t, 1 + t, t + t^2 + t^4]$

$= \frac{1+t^2+t^4+t^6+t^7+t^8+t^9}{t^7+t^9+t^{10}+t^{12}}$

$[0, \overrightarrow{X_4}] + [0, \overleftarrow{X_4}] = \frac{1+t^2+t^8}{t^7+t^9+t^{10}+t^{12}} + \frac{1+t^2+t^4+t^6+t^7+t^8+t^9}{t^7+t^9+t^{10}+t^{12}} = \frac{t^4+t^6+t^7+t^9}{t^7+t^9+t^{10}+t^{12}}$

$= \frac{1}{t^3}$, which corroborates the stated claim.

$q_{k_4} = t^4 d_2 = t^7 + t^9 + t^{10} + t^{12}$ and $p_{k_4} = d_2(1 + \frac{1}{d_1} + \frac{1}{d_2}) = d_2 + \frac{d_2}{d_1} + \frac{d_2}{d_2}$

$= 1 + t^2 + t^8$.

$Q = td_2 + p_{k_4} = 1 + t^2 + t^4 + t^6 + t^7 + t^8 + t^9$.

$P = \frac{1+p_{k_4}Q}{q_{k_4}} = \frac{1+(1+t^2+t^8)(1+t^2+t^4+t^6+t^7+t^8+t^9)}{t^7+t^9+t^{10}+t^{12}}$. Now,

$P = \frac{1+p_{k_4}^2+td_2p_{k_4}}{t^4d_2} = \frac{1+(d_2+\frac{d_2}{d_1}+\frac{d_2}{d_2})^2+td_2(d_2+\frac{d_2}{d_1}+\frac{d_2}{d_2})}{t^4d_2} = \frac{d_2^2+d_1^2d_2^2+d_1^2d_2t+d_1d_2^2t+d_1^2d_2^2t}{t^4d_1^2d_2}$

$= 1 + t + t^2 + t^4 + t^5$

It is seen that P is an integer.

$Pq_{k_4} - Qp_{k_4} = (1 + t + t^2 + t^4 + t^5)(t^7 + t^9 + t^{10} + t^{12})$

$- (1 + t^2 + t^8)(1 + t^2 + t^4 + t^6 + t^7 + t^8 + t^9) = 1$

For $n = 4$, $td_2 = t^4 + t^6 + t^7 + t^9 \equiv 0 \pmod{t^4}$ and

$\frac{d_2}{d_{4-i}^2} + t\frac{d_2}{d_{4-i+1}} = \frac{d_2}{d_{4-i+1}}([4 - i + 1] + t) = \frac{d_2}{d_{4-i+1}} t^{2^{4-i+1}} \equiv 0 \pmod{t^n}$

As an example, if $i = 3$, then,

$\frac{d_2}{d_1^2} + t\frac{d_2}{d_2} = \frac{d_2}{d_2}([2] + t) = \frac{d_2}{d_2} t^{2^2} = t^4 \equiv 0 \pmod{t^4}$

By continued fraction fact 1.1.3, (Chapter 2.2.1, p. 10), $p_{k_4-1}q_{k_4} - q_{k_4-1}p_{k_4} = 1$, so that $p_{k_4}(q_{k_4-1} - Q) = q_{k_4}(p_{k_4-1} - P)$. The degree of $p_{k_4-1} - P$ is less than that of p_{k_4}. Hence, $P = p_{k_4-1}$.

Appendix G

THE EXPONENTIAL AND THE DEGREE TWO PRIME

Examine Dr. Thakur's development of the CF for $e/\overline{p}$, where $\overline{p} = t^2 + t + 1$, the degree two prime. He details this with Theorem 5 (Thakur, 1996, p. 257).

Let $\theta_i := \sum_{j=0}^{i} \frac{1}{\overline{p} d_j}$. Then,

$\theta_0 := \frac{1}{t^2+t+1}$

$\mu_0 = [0,\, t^2 + t + 1]$

$\theta_1 := \frac{1}{(t^2+t+1)d_0} + \frac{1}{(t^2+t+1)d_1} = \frac{1}{t^2+t+1} + \frac{1}{(t^2+t+1)[1]}$

This two term partial sum can be converted to a simple continued fraction using the "Rational Function to Simple CF Expansion Algorithm," found in Appendix 3, p. 80.

Observe exactly how this is done.

$\frac{1}{t^2+t+1} + \frac{1}{(t^2+t+1)[1]} = \frac{1}{t^2+t} = \frac{1}{[1]}$

$\mu_1 = [0,\, [1]]$

$\theta_2 := \frac{1}{(t^2+t+1)d_0} + \frac{1}{(t^2+t+1)d_1} + \frac{1}{(t^2+t+1)d_2}$

$\quad = \frac{1}{t^2+t+1} + \frac{1}{(t^2+t+1)[1]} + \frac{1}{(t^2+t+1)[2][1]^2} = \frac{1+t^2+t^8}{t^3+t^4+t^9+t^{10}}$

(Here, the polynomials in t are displayed in ascending exponent order to correspond with Mathematica 5.2 output.)

First, consider the expression $\frac{1+t^2+t^8}{t^3+t^4+t^9+t^{10}}$. The first step is to convert it to μ_2, a simple continued fraction comprised of polynomials in t.

$pq_0(\theta_2) = 0$, $pr_0(\theta_2) = 1 + t^2 + t^8$, $pd_0(\theta_2) = t^3 + t^4 + t^9 + t^{10}$

Now consider $\frac{pd_0}{pr_0} = \frac{t^3+t^4+t^9+t^{10}}{1+t^2+t^8}$

$pq_1(\theta_2) = [1]$, $pr_1(\theta_2) = [1]$, $pd_1(\theta_2) = 1 + t^2 + t^8$

Now consider $\frac{pd_1}{pr_1} = \frac{1+t^2+t^8}{[1]}$

$pq_2(\theta_2) = t + t^2 + t^3 + t^4 + t^5 + t^6$, $pr_2(\theta_2) = 1$, $pd_2(\theta_2) = [1]$

Now consider $\frac{pd_2}{pr_2} = \frac{[1]}{1}$

$pq_3(\theta_2) = [1]$, $pr_3(\theta_2) = 0$, $pd_3(\theta_2) = 1$

The evaluation of θ_2 is done at this point. If the pq_i terms are strung together, $i \rightarrow 0$ to 3, the simple continued fraction expression for μ_2 is obtained, below.

$\mu_2 = [0, [1], t + t^2 + t^3 + t^4 + t^5 + t^6, [1]]$

Denote μ_2 in "Curly Bracket" notation as defined in Appendix C.1 on page 73. The following is seen:

$\mu_2 = [0, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}]$

$$\theta_3 := \frac{1}{(t^2+t+1)d_0} + \frac{1}{(t^2+t+1)d_1} + \frac{1}{(t^2+t+1)d_2} + \frac{1}{(t^2+t+1)d_3}$$
$$= \frac{1}{t^2+t+1} + \frac{1}{(t^2+t+1)[1]} + \frac{1}{(t^2+t+1)[2][1]^2} + \frac{1}{(t^2+t+1)[3][2]^2[1]^4}$$

The same process used to convert θ_2 to μ_2, above, is used to evaluate all subsequent μ_i terms.

$\mu_3 := [0, [1], [2]\,\overline{p}, [1], [2][1] + 1, [2], [1], [2]]$

$= [0, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{0, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\}]$

$\mu_4 := [0, [1], [2]\,\overline{p}, [1], [2][1] + 1, [2], [1], [2], \frac{[4]}{\overline{p}}, [2], [1], [2], [2][1] + 1, [1], [2]\,\overline{p}, [1]]$

$= [0, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{0, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\},$
$\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\},$
$\{1, 4\}, \{1, 2\}, \{1, 4\}, \{0, 2, 3, 5, 6\}, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}]$

$\mu_5 := [0, [1], [2]\,\overline{p}, [1], [2][1] + 1, [2], [1], [2], \frac{[4]}{\overline{p}}, [2], [1], [2], [2][1] + 1, [1], [2]\,\overline{p}, [1],$
$\lfloor\frac{[5]}{\overline{p}}\rfloor, [2], [1], [2], [2][1] + 1, [1], [2]\,\overline{p}, [1], \frac{[4]}{\overline{p}}, [1], [2]\,\overline{p}, [1], [2][1] + 1, [2], [1], [2]]$

where $\lfloor\frac{[5]}{\overline{p}}\rfloor$ denotes the polynomial obtained as the quotient when [5] is divided by $\overline{p}$ using the division algorithm.

$\lfloor\frac{[5]}{\overline{p}}\rfloor = \{0, 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27, 29, 30\}$

$\mu_5 := [0, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{0, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\},$
$\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\},$
$\{1, 4\}, \{1, 2\}, \{1, 4\}, \{0, 2, 3, 5, 6\}, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\},$

{0, 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27, 29, 30},
{1, 4}, {1, 2}, {1, 4}, {0, 2, 3, 5, 6}, {1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2},
{1, 2, 4, 5, 7, 8, 10, 11, 13, 14},
{1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2}, {0, 2, 3, 5, 6}, {1, 4}, {1, 2}, {1, 4}]

Continuing with "Curly Bracket" notation, the expansion of the μ_7 is as follows:

μ_7 := [0, {1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2}, {0, 2, 3, 5, 6}, {1, 4}, {1, 2}, {1, 4},
{1, 2, 4, 5, 7, 8, 10, 11, 13, 14},
{1, 4}, {1, 2}, {1, 4}, {0, 2, 3, 5, 6}, {1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2},
{0, 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27, 29, 30},
{1, 4}, {1, 2}, {1, 4}, {0, 2, 3, 5, 6}, {1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2},
{1, 2, 4, 5, 7, 8, 10, 11, 13, 14},
{1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2}, {0, 2, 3, 5, 6}, {1, 4}, {1, 2}, {1, 4},
{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26, 28, 29, 31, 32, 34, 35, 37, 38, 40, 41, 43, 44, 46, 47, 49, 50, 52, 53, 55, 56, 58, 59, 61, 62},
{1, 4}, {1, 2}, {1, 4}, {0, 2, 3, 5, 6}, {1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2},
{1, 2, 4, 5, 7, 8, 10, 11, 13, 14},
{1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2}, {0, 2, 3, 5, 6}, {1, 4}, {1, 2}, {1, 4},
{0, 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27, 29, 30},
{1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2}, {0, 2, 3, 5, 6}, {1, 4}, {1, 2}, {1, 4},
{1, 2, 4, 5, 7, 8, 10, 11, 13, 14},
{1, 4}, {1, 2}, {1, 4}, {0, 2, 3, 5, 6}, {1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2},
{0, 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27, 29, 30, 32, 33, 35, 36, 38, 39, 41, 42, 44, 45, 47, 48, 50, 51, 53, 54, 56, 57, 59, 60, 62, 63, 65, 66, 68, 69, 71, 72, 74, 75, 77, 78, 80, 81, 83, 84, 86, 87, 89, 90, 92, 93, 95, 96, 98, 99, 101, 102, 104, 105, 107, 108, 110, 111, 113, 114, 116, 117, 119, 120, 122, 123, 125, 126},
{1, 4}, {1, 2}, {1, 4}, {0, 2, 3, 5, 6}, {1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2},
{1, 2, 4, 5, 7, 8, 10, 11, 13, 14},
{1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2}, {0, 2, 3, 5, 6}, {1, 4}, {1, 2}, {1, 4},
{0, 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27, 29, 30},
{1, 2}, {1, 2, 3, 4, 5, 6}, {1, 2}, {0, 2, 3, 5, 6}, {1, 4}, {1, 2}, {1, 4},
{1, 2, 4, 5, 7, 8, 10, 11, 13, 14},

$\{1, 4\}, \{1, 2\}, \{1, 4\}, \{0, 2, 3, 5, 6\}, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\},$

$\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26, 28, 29, 31, 32, 34, 35, 37, 38, 40, 41, 43, 44, 46, 47, 49, 50, 52, 53, 55, 56, 58, 59, 61, 62\},$

$\{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{0, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\},$

$\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\},$

$\{1, 4\}, \{1, 2\}, \{1, 4\}, \{0, 2, 3, 5, 6\}, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\},$

$\{0, 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27, 29, 30\},$

$\{1, 4\}, \{1, 2\}, \{1, 4\}, \{0, 2, 3, 5, 6\}, \{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\},$

$\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\},$

$\{1, 2\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2\}, \{0, 2, 3, 5, 6\}, \{1, 4\}, \{1, 2\}, \{1, 4\}]$

Appendix H

COMPUTATIONAL CORROBORATION OF SECTION 5 EXAMPLE IN 1997 ARTICLE

Time did not permit that this thesis would include a study of Dr. Thakur's third article related to exponential and continued fractions, "Patterns of Continued Fractions for the Analogues of e and Related Numbers in the Function Field Case," (Thakur, 1997).

A strong focus and interest in writing this thesis was to be able to reproduce the sequence expansion of $\alpha = \frac{e}{t^3+t+1}$, as seen in Section 5, p. 143 of the 1997 article.

The Mathematica 5.2 program outlined in Appendix I, p. 113, and "Curly Bracket" expanded notation described in Appendix C.1, p. 73 were used to generate and list the following sequence.

The first 420 terms of the CF expansion:

$\frac{e}{t^3+t+1}$ = [0, {3}, {0, 1}, {1}, {1}, {1, 3, 4, 5}, {1, 2, 3, 5}, {1, 3, 4, 5}, {1}, {1}, {0, 1}, {3}, {2, 3, 4, 6, 9, 10, 11, 13}, {0, 1}, {0, 1}, {0, 1, 2}, {1}, {1}, {3}, {0, 1}, {1}, {1, 2, 3, 5}, {1}, {0, 1}, {3}, {1}, {1}, {0, 1, 2}, {0, 1}, {0, 1},

{1, 4, 5, 6, 8, 11, 12, 13, 15, 18, 19, 20, 22, 25, 26, 27, 29},

{1, 2}, {1, 2, 5, 7}, {1, 2}, {1, 2, 3, 5}, {1, 2}, {1, 2, 5, 7}, {1, 2}, {2, 3, 4, 6, 9, 10, 11, 13}, {1}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1, 2}, {1, 2, 4, 8, 9, 11}, {1, 2}, {1}, {0, 1}, {0, 1}, {0, 1, 2}, {1},

{1, 2, 3, 5, 8, 9, 10, 12, 15, 16, 17, 19, 22, 23, 24, 26, 29, 30, 31, 33, 36, 37, 38, 40, 43, 44, 45, 47, 50, 51, 52, 54, 57, 58, 59, 61},

{1}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1, 2}, {1, 2, 4, 8, 9, 11}, {1, 2}, {1}, {0, 1}, {0, 1}, {0, 1, 2}, {1}, {2, 3, 4, 6, 9, 10, 11, 13}, {1, 2}, {1, 2, 5, 7}, {1, 2}, {1, 2, 3, 5}, {1, 2}, {1, 2, 5, 7},

{1, 2}, {1, 4, 5, 6, 8, 11, 12, 13, 15, 18, 19, 20, 22, 25, 26, 27, 29},

{0, 1}, {0, 1}, {0, 1, 2}, {1}, {1}, {3}, {0, 1}, {1}, {1, 2, 3, 5}, {1}, {0, 1}, {3}, {1}, {1}, {0, 1, 2}, {0, 1}, {0, 1}, {2, 3, 4, 6, 9, 10, 11, 13}, {3}, {0, 1}, {1}, {1}, {1, 3, 4, 5}, {1, 2, 3, 5}, {1, 3, 4, 5}, {1}, {1}, {0, 1}, {3}, {2, 3, 4, 6, 9, 10, 11, 13, 16, 17, 18, 20, 23, 24, 25, 27, 30, 31, 32, 34, 37, 38, 39, 41, 44, 45, 46, 48, 51, 52, 53, 55, 58, 59, 60, 62, 65, 66, 67, 69, 72, 73, 74, 76, 79, 80, 81, 83, 86, 87, 88, 90, 93, 94, 95, 97, 100, 101, 102, 104, 107, 108, 109, 111, 114, 115, 116, 118, 121, 122, 123, 125},

{0, 1}, {0, 1}, {0, 1, 2}, {1}, {1}, {3}, {0, 1}, {1}, {1, 2, 3, 5}, {1}, {0, 1}, {3}, {1}, {1}, {0, 1, 2}, {0, 1}, {0, 1}, {2, 3, 4, 6, 9, 10, 11, 13}, {3}, {0, 1}, {1}, {1}, {1, 3, 4, 5}, {1, 2, 3, 5}, {1, 3, 4, 5}, {1}, {1}, {0, 1}, {3},

{1, 4, 5, 6, 8, 11, 12, 13, 15, 18, 19, 20, 22, 25, 26, 27, 29},

{1}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1, 2}, {1, 2, 4, 8, 9, 11}, {1, 2}, {1}, {0, 1}, {0, 1}, {0, 1, 2}, {1}, {2, 3, 4, 6, 9, 10, 11, 13}, {1, 2}, {1, 2, 5, 7}, {1, 2}, {1, 2, 3, 5}, {1, 2}, {1, 2, 5, 7}, {1, 2},

{1, 2, 3, 5, 8, 9, 10, 12, 15, 16, 17, 19, 22, 23, 24, 26, 29, 30, 31, 33, 36, 37, 38, 40, 43, 44, 45, 47, 50, 51, 52, 54, 57, 58, 59, 61},

{1, 2}, {1, 2, 5, 7}, {1, 2}, {1, 2, 3, 5}, {1, 2}, {1, 2, 5, 7}, {1, 2}, {2, 3, 4, 6, 9, 10, 11, 13}, {1}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1, 2}, {1, 2, 4, 8, 9, 11}, {1, 2}, {1}, {0, 1}, {0, 1}, {0, 1, 2}, {1},

{1, 4, 5, 6, 8, 11, 12, 13, 15, 18, 19, 20, 22, 25, 26, 27, 29},

{3}, {0, 1}, {1}, {1}, {1, 3, 4, 5}, {1, 2, 3, 5}, {1, 3, 4, 5}, {1}, {1}, {0, 1}, {3}, {2, 3, 4, 6, 9, 10, 11, 13}, {0, 1}, {0, 1}, {0, 1, 2}, {1}, {1}, {3}, {0, 1}, {1}, {1, 2, 3, 5}, {1}, {0, 1}, {3}, {1}, {1}, {0, 1, 2}, {0, 1}, {0, 1},

{1, 4, 5, 6, 8, 11, 12, 13, 15, 18, 19, 20, 22, 25, 26, 27, 29, 32, 33, 34, 36, 39, 40, 41, 43, 46, 47, 48, 50, 53, 54, 55, 57, 60, 61, 62, 64, 67, 68, 69, 71, 74, 75, 76, 78, 81, 82, 83, 85, 88, 89, 90, 92, 95, 96, 97, 99, 102, 103, 104, 106, 109, 110, 111, 113, 116, 117, 118, 120, 123, 124, 125, 127, 130, 131, 132, 134, 137, 138, 139, 141, 144, 145, 146, 148, 151, 152, 153, 155, 158, 159, 160, 162, 165, 166, 167, 169, 172, 173, 174, 176, 179, 180, 181, 183, 186, 187, 188, 190, 193, 194, 195, 197, 200, 201, 202, 204, 207, 208, 209, 211, 214, 215, 216, 218, 221, 222, 223, 225, 228, 229, 230, 232, 235, 236, 237, 239, 242, 243, 244, 246, 249, 250, 251, 253},

{1, 2}, {1, 2, 5, 7}, {1, 2}, {1, 2, 3, 5}, {1, 2}, {1, 2, 5, 7}, {1, 2}, {2, 3, 4, 6, 9, 10, 11, 13}, {1}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1, 2}, {1, 2, 4, 8, 9, 11}, {1, 2}, {1}, {0, 1}, {0, 1}, {0, 1, 2}, {1},

{1, 4, 5, 6, 8, 11, 12, 13, 15, 18, 19, 20, 22, 25, 26, 27, 29},

{3}, {0, 1}, {1}, {1}, {1, 3, 4, 5}, {1, 2, 3, 5}, {1, 3, 4, 5}, {1}, {1}, {0, 1}, {3}, {2, 3, 4, 6, 9, 10, 11, 13}, {0, 1}, {0, 1}, {0, 1, 2}, {1}, {1}, {3}, {0, 1}, {1}, {1, 2, 3, 5}, {1}, {0, 1}, {3}, {1}, {1}, {0, 1, 2}, {0, 1}, {0, 1},

{1, 2, 3, 5, 8, 9, 10, 12, 15, 16, 17, 19, 22, 23, 24, 26, 29, 30, 31, 33, 36, 37, 38, 40, 43, 44, 45, 47, 50, 51, 52, 54, 57, 58, 59, 61},

{0, 1}, {0, 1}, {0, 1, 2}, {1}, {1}, {3}, {0, 1}, {1}, {1, 2, 3, 5}, {1}, {0, 1}, {3}, {1}, {1}, {0, 1, 2}, {0, 1}, {0, 1}, {2, 3, 4, 6, 9, 10, 11, 13}, {3}, {0, 1}, {1}, {1}, {1, 3, 4, 5}, {1, 2, 3, 5}, {1, 3, 4, 5}, {1}, {1}, {0, 1}, {3},

{1, 4, 5, 6, 8, 11, 12, 13, 15, 18, 19, 20, 22, 25, 26, 27, 29},

{1}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1, 2}, {1, 2, 4, 8, 9, 11}, {1, 2}, {1}, {0, 1}, {0, 1}, {0, 1, 2}, {1}, {2, 3, 4, 6, 9, 10, 11, 13}, {1, 2}, {1, 2, 5, 7}, {1, 2}, {1, 2, 3, 5}, {1, 2}, {1, 2, 5, 7},

{1, 2}, {2, 3, 4, 6, 9, 10, 11, 13, 16, 17, 18, 20, 23, 24, 25, 27, 30, 31, 32, 34, 37, 38, 39, 41, 44, 45, 46, 48, 51, 52, 53, 55, 58, 59, 60, 62, 65, 66, 67, 69, 72, 73, 74, 76, 79, 80, 81, 83, 86, 87, 88, 90, 93, 94, 95, 97, 100, 101, 102, 104, 107, 108, 109, 111, 114, 115, 116, 118, 121, 122, 123, 125},

{1}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1, 2}, {1, 2, 4, 8, 9, 11}, {1, 2}, {1}, {0, 1}, {0, 1}, {0, 1, 2}, {1}, {2, 3, 4, 6, 9, 10, 11, 13}, {1, 2}, {1, 2, 5, 7}, {1, 2}, {1, 2, 3, 5}, {1, 2}, {1, 2, 5, 7}, {1, 2},

{1, 4, 5, 6, 8, 11, 12, 13, 15, 18, 19, 20, 22, 25, 26, 27, 29},

{0, 1}, {0, 1}, {0, 1, 2}, {1}, {1}, {3}, {0, 1}, {1}, {1, 2, 3, 5}, {1}, {0, 1}, {3}, {1}, {1}, {0, 1, 2}, {0, 1}, {0, 1}, {2, 3, 4, 6, 9, 10, 11, 13}, {3}, {0, 1}, {1}, {1}, {1, 3, 4, 5}, {1, 2, 3, 5}, {1, 3, 4, 5}, {1}, {1}, {0, 1}, {3},

{1, 2, 3, 5, 8, 9, 10, 12, 15, 16, 17, 19, 22, 23, 24, 26, 29, 30, 31, 33, 36, 37, 38, 40, 43, 44, 45, 47, 50, 51, 52, 54, 57, 58, 59, 61},

{3}, {0, 1}, {1}, {1}, {1, 3, 4, 5}, {1, 2, 3, 5}, {1, 3, 4, 5}, {1}, {1}, {0, 1}, {3}, {2, 3, 4, 6, 9, 10, 11, 13}, {0, 1}, {0, 1}, {0, 1, 2}, {1}, {1}, {3}, {0, 1}, {1}, {1, 2, 3, 5}, {1}, {0, 1}, {3}, {1}, {1}, {0, 1, 2}, {0, 1}, {0, 1},

{1, 4, 5, 6, 8, 11, 12, 13, 15, 18, 19, 20, 22, 25, 26, 27, 29},

{1, 2}, {1, 2, 5, 7}, {1, 2}, {1, 2, 3, 5}, {1, 2}, {1, 2, 5, 7}, {1, 2}, {2, 3, 4, 6, 9, 10, 11, 13}, {1}, {0, 1, 2}, {0, 1}, {0, 1}, {1}, {1, 2}, {1, 2, 4, 8, 9, 11}, {1, 2}, {1}, {0, 1}, {0, 1}, {0, 1, 2}, {1},

{1, 2, 3, 5, 8, 9, 10, 12, 15, 16, 17, 19, 22, 23, 24, 26, 29, 30, 31, 33, 36, 37, 38, 40, 43, 44, 45, 47, 50, 51, 52, 54, 57, 58, 59, 61, 64, 65, 66, 68, 71, 72, 73, 75, 78, 79, 80, 82, 85, 86, 87, 89, 92, 93, 94, 96, 99, 100, 101, 103, 106, 107, 108, 110, 113, 114, 115, 117, 120, 121, 122, 124, 127, 128, 129, 131, 134, 135, 136, 138, 141, 142, 143, 145, 148, 149, 150, 152, 155, 156, 157, 159, 162, 163, 164, 166, 169, 170, 171, 173, 176, 177, 178, 180, 183, 184, 185, 187, 190, 191, 192, 194, 197, 198, 199, 201, 204, 205, 206, 208, 211, 212, 213, 215, 218, 219, 220, 222, 225, 226, 227, 229, 232, 233, 234, 236, 239, 240, 241, 243, 246, 247, 248, 250, 253, 254, 255, 257, 260, 261, 262, 264, 267, 268, 269, 271, 274, 275, 276, 278, 281, 282, 283, 285, 288, 289, 290, 292, 295, 296, 297, 299, 302, 303, 304, 306, 309, 310, 311, 313, 316, 317, 318, 320, 323, 324, 325, 327, 330, 331, 332, 334, 337, 338, 339, 341, 344, 345, 346, 348, 351, 352, 353, 355, 358, 359, 360, 362, 365, 366, 367, 369, 372, 373, 374, 376, 379, 380, 381, 383, 386, 387, 388, 390, 393, 394, 395, 397, 400, 401, 402, 404, 407, 408, 409, 411, 414, 415, 416, 418, 421, 422, 423, 425, 428, 429, 430, 432, 435, 436, 437, 439, 442, 443, 444, 446, 449, 450, 451, 453, 456, 457, 458, 460, 463, 464, 465, 467, 470, 471, 472, 474, 477, 478, 479, 481, 484, 485, 486, 488, 491, 492, 493, 495, 498, 499, 500, 502, 505, 506, 507, 509},

{1}, {0, 1, 2}, {0, 1}, ...]

Appendix I

OCTOBER 2006 MATHEMATICA 5.2 PROGRAM TO EVALUATE SIMPLE AND GENERAL HURWITZ NUMBERS

A variation of this program was used to generate all of the simple and general Hurwitz number continued fraction expansions listed in this thesis. It is referenced in Appendix E.2, p. 80, and in Appendix H, p. 109.

This particular program will evaluate $e(t) + t$, for $q = 2$. The results can be seen in Appendix E.3.2, p. 82

```
Date[]
{2006, 10, 17, 11, 49, 40.5725568}
ClearAll["Global`*"]

q = 2; z = t;
f[i_] = t^q^i - t
d[0] := 1
d[1] := f[1]

Table[{PlaceCounter = 1; d[i] = f[PlaceCounter]^q^(i - 1);
     Do[{d[i] = f[PlaceCounter + 1]^q^(i - 1 - PlaceCounter)
     d[i]; PlaceCounter += 1}, {i - 1}]}, {i, 2, 50}];

x[n_] := PolynomialMod[z/1 + Sum[z^q^j/d[j], {j, 1, n}], q]

a[0] = PolynomialMod[x[8] - t, q];
Hurwitz = PadRight[{a[0]}, 500];
Take[Hurwitz, 100];
temp = Hurwitz;
pq = PadRight[{0}, 500];
pr = PadRight[{0}, 500];

pq[[1]] = PolynomialMod[PolynomialQuotient
     [Numerator[temp[[1]]], Denominator[temp[[1]]], t], q];
```

```
pr[[1]] = PolynomialMod[PolynomialRemainder
     [Numerator[temp[[1]]], Denominator[temp[[1]]], t], q];
temp[[2]] = Together[Denominator[temp[[1]]]/pr[[1]]];

pq[[2]] = PolynomialMod[PolynomialQuotient
     [Numerator[temp[[2]]], Denominator[temp[[2]]], t], q];
pr[[2]] = PolynomialMod[PolynomialRemainder
     [Numerator[temp[[2]]], Denominator[temp[[2]]], t], q];
temp[[3]] = Together[Denominator[temp[[2]]]/pr[[2]]];

pq[[3]] = PolynomialMod[PolynomialQuotient
     [Numerator[temp[[3]]], Denominator[temp[[3]]], t], q];
pr[[3]] = PolynomialMod[PolynomialRemainder
     [Numerator[temp[[3]]], Denominator[temp[[3]]], t], q];
temp[[4]] = Together[Denominator[temp[[3]]]/pr[[3]]];

Table[{pq[[n]] = PolynomialMod[PolynomialQuotient
     [Numerator[temp[[n]]], Denominator[temp[[n]]], t], q],
     pr[[n]] = PolynomialMod[PolynomialRemainder
     [Numerator[temp[[n]]], Denominator[temp[[n]]], t], q],
     temp[[n + 1]] = Together[Denominator[temp[[n]]]/pr[[n]]]},
     {n, 4, 70}];

Take[pq, 65]

InputForm[%]
```

Appendix J

DECEMBER 1976 FORTRAN PROGRAM TO CALCULATE EULER'S NUMBER

This Fortran program was written in 1976, just after the author of this thesis graduated college with a BA in mathematics. At the time, she was working for General Electric Corporation as an engineering assistant. This program calculates e to 786 places, which was a detailed expansion for the time.

(Note that the column alignment is representational.)

```
      DIMENSION BLOCK(168), SMBLK(168)
      INTEGER BLOCK,DIV,QUO,REM,SMBLK
      DO 33   J = 1,168
      BLOCK(J) = 0
   33 SMBLK(J) = 0
      BLOCK(I) = 500000
      DO 1    DIV = 3,369
      DO 2    I = 1,167
      IP = I + 1
      QUO = BLOCK(I)/DIV
      REM = BLOCK(I) − DIV * QUO
      BLOCK(I) = QUO
      BLOCK(IP) = REM * 1000000 + BLOCK(IP)
    2 CONTINUE
      DO 3    I = 1,167
      J = 168 − I
      JM = J − 1
      SMBLK(J) = SMBLK(J) + BLOCK(J)
      IF (SMBLK(J) .LE. 999999)     GO TO 4
      SMBLK(J) = SMBLK(J) − 1000000
      SMBLK(JM) = SMBLK(JM) + 1
    4 CONTINUE
    3 CONTINUE
    1 CONTINUE
```

```
      PRINT 5,(BLOCK(I),I = 1,167)
      PRINT 5,(SMBLK(I),I = 1,167)
    5 FORMAT(1X,10I7)
      STOP
      END
```

This program approximated e with the following method.

First, note that $e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots$

$= 1 + 1 + \frac{1}{2} + \frac{\frac{1}{2}}{3} + \frac{\frac{\frac{1}{2}}{3}}{4} + \frac{\frac{\frac{\frac{1}{2}}{3}}{4}}{5} + \cdots$

The program iteratively calculates the terms and partial sums of the above expansion, so that:

$\text{BLOCK}_0 = \frac{1}{2} =$ [500000 000000 000000 000000 000000 000000 ...] for 168 terms.

$\text{BLOCK}_1 = \frac{\frac{1}{2}}{3} =$ [166666 666666 666666 666666 666666 666666 ...] for 168 terms.

SMBLK was initialized to BLOCK at this point:

$\text{SMBLK}_1 = \text{BLOCK}_1 =$ [166666 666666 666666 666666 666666 666666 ...] for 168 terms.

BLOCK is then divided iteratively by 4, 5, 6, etc. to 369. SMBLK is then the partial sum of each iteration as follows:

$\text{BLOCK}_2 = \text{BLOCK}_1/4 = \frac{\frac{\frac{1}{2}}{3}}{4}$ and

$\text{SMBLK}_2 = \text{SMBLK}_1 + \text{BLOCK}_2 = \frac{\frac{1}{2}}{3} + \frac{\frac{\frac{1}{2}}{3}}{4}$

$\text{BLOCK}_2 =$ [041666 666666 666666 666666 666666 666666 ...] for 168 terms.

$\text{SMBLK}_2 =$ [208333 333333 333333 333333 333333 333333 ...] for 168 terms.

$\text{BLOCK}_3 = \text{BLOCK}_2/5 = \frac{\frac{\frac{\frac{1}{2}}{3}}{4}}{5}$ and

$\text{SMBLK}_3 = \text{SMBLK}_2 + \text{BLOCK}_3 = \frac{\frac{1}{2}}{3} + \frac{\frac{\frac{1}{2}}{3}}{4} + \frac{\frac{\frac{\frac{1}{2}}{3}}{4}}{5}$

$\text{BLOCK}_3 =$ [008333 333333 333333 333333 333333 333333 ...]

$\text{SMBLK}_3 =$ [216666 666666 666666 666666 666666 666666 ...]

$\text{BLOCK}_4 = \text{BLOCK}_3/6 = \frac{\frac{\frac{\frac{\frac{1}{2}}{3}}{4}}{5}}{6}$ and

$\text{SMBLK}_4 = \text{SMBLK}_3 + \text{BLOCK}_4 = \frac{\frac{1}{2}}{3} + \frac{\frac{\frac{1}{2}}{3}}{4} + \frac{\frac{\frac{\frac{1}{2}}{3}}{4}}{5} + \frac{\frac{\frac{\frac{\frac{1}{2}}{3}}{4}}{5}}{6}$

BLOCK_4 = [001388 888888 888888 888888 888888 888888 ...]

SMBLK_4 = [218055 555555 555555 555555 555555 555555 ...]

$\text{BLOCK}_5 = \text{BLOCK}_4/7 = \frac{\frac{\frac{\frac{\frac{\frac{1}{2}}{3}}{4}}{5}}{6}}{7}$

and $\text{SMBLK}_5 = \text{SMBLK}_4 + \text{BLOCK}_5 = \frac{\frac{1}{2}}{3} + \frac{\frac{\frac{1}{2}}{3}}{4} + \frac{\frac{\frac{\frac{1}{2}}{3}}{4}}{5} + \frac{\frac{\frac{\frac{\frac{1}{2}}{3}}{4}}{5}}{6} + \frac{\frac{\frac{\frac{\frac{\frac{1}{2}}{3}}{4}}{5}}{6}}{7}$

BLOCK_5 = [000198 412698 412698 412698 412698 412698 ...]

SMBLK_5 = [218253 968253 968253 968253 968253 968253 ...]

The BLOCK array approaches 0, and the SMBLK array approaches the decimal expansion of $e - 2.5$. After 369 iterations, SMBLK accurately replicates $(e - 2.5)$ to 768 digits as:

SMBLK_{369} = [218281 828459 045235 360287 471352 662497 757247 093699 959574 966967 627724 076630 353547 594571 382178 525166 427427 466391 932003 059921 817413 596629 043572 900334 295260 595630 738132 328627 943490 763233 829880 753195 251019 011573 834187 930702 154089 149934 884167 509244 761460 668082 264800 168477 411853 742345 442437 107539 077744 992069 551702 761838 606261 331384 583000 752044 933826 560297 606737 113200 709328 709127 443747 047230 696977 209310 141692 836819 025515 108657 463772 111252 389784 425056 953696 770785 449969 967946 864454 905987 931636 889230 098793 127736 178215 424999 229576 351482 208269 895193 668033 182528 869398 496465 105820 939239 829488 793320 362509 443117 301238 197068 416140 397019 837679 320683 282376 464804 295311 802328 782509 819455 815301 756717 361332 069811 250996 181881 593041 690351 598888 519345 807273 866738 589422 879228 499892 086805 825749 279610 484198 44]

Then, $e \approx 2.5 +$ .218281 828459 045235 360287 471352 662497 757247 093699 959574 966967 627724 076630 353547 594571 382178 525166 427427 ...

$\approx$ 2.718281 828459 045235 360287 471352 662497 757247 093699 959574 966967 627724 076630 353547 594571 382178 525166 427427 ...

Appendix K

THE RULER FUNCTION

The sequence expansion of the Ruler Function is included here, because the $e(z)$ function for $q = 2$ corresponds to it exactly. To the author, this speaks to the inherent mathematical fundamental and cohesive properties of the exponential function. Dr. Thakur's articles have brought this to light.

Dr. Thakur is referenced on the web page, on the last line of this listing.

The following information on the Ruler Function was found May 6, 2007 on:

N. J. A. Sloane, (2007), The On-Line Encyclopedia of Integer Sequences, published electronically at www.research.att.com/ njas/sequences/.

%I A001511 M0127 N0051

%S A001511 1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,5,1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,6,1,2,1,

%T A001511 3,1,2,1,4,1,2,1,3,1,2,1,5,1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,7,1,2,1,3,1,2,

%U A001511 1,4,1,2,1,3,1,2,1,5,1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,6,1,2,1,3,1,2,1,4,1

%N A001511 The ruler function: 2ˆa(n) divides 2n. Or, a(n) = 2–adic valuation of 2n.

%C A001511 a(n) is the number of digits that must be counted from right to left to reach the first 1 in the binary representation of n. For example, a(12)=3 digits must be counted from right to left to reach the first 1 in 1100, the binary representation of 12. –anon, May 17 2002

%C A001511 If you are counting in binary, and the least significant bit is numbered 1, the next bit is 2, etc., a(n) is the bit that is incremented when increasing from n–1 to n. –Jud McCranie, Apr 26, 2004

%C A001511 Number of steps to reach an integer starting with (n+1)/2 and using the map x –>x*ceiling(x) (cf. A073524).

%C A001511 a(n) = number of disk to be moved at n–th step of optimal solution to Tower of Hanoi problem (comment from Andreas M. Hinz (hinz(AT)appl–math.tu–muenchen.de)).

%C A001511 Shows which bit to flip when creating the binary reflected Gray code (bits are numbered from the right, offset is 1). This is essentially equivalent to Hinz's comment. –Adam Kertesz (adamkertesz(AT)worldnet.att.net), Jul 28 2001

%C A001511 a(n) is the Hamming distance between n and n–1 (in binary). This is equivalent to Kertesz's comments above. –Tak–Shing Chan (chan12(AT)alumni.usc.edu), Feb 25 2003

%C A001511 Let S(0) = {1}, S(n) = {S(n–1), S(n–1)–{x}, x+1} where x = last term of S(n–1); sequence gives S(infinity). –Benoit Cloitre (abmt(AT)wanadoo.fr), Jun 14 2003

%C A001511 The sum of all terms up to and including the first occurrence of m is 2^m–1. –Donald Sampson (marsquo(AT)hotmail.com), Dec 01 2003

%C A001511 m appears every 2^m terms starting with the 2^(m–1)th term. –Donald Sampson (marsquo(AT)hotmail.com), Dec 08 2003

%C A001511 Sequence read mod 4 gives A092412. –DELEHAM Philippe (kolotoko(AT)wanadoo.fr), Mar 28 2004

%C A001511 If q = 2n/2^A001511(n) and if b(m) is defined by b(0)=q–1 and b(m)=2*b(m–1)+1, then 2n = b(A001511(n)) + 1. –Gerald McGarvey (Gerald.McGarvey(AT)comcast.net), Dec 18 2004

%C A001511 Repeating pattern ABACABADABACABAE ... –Jeremy Gardiner (jeremy.gardiner(AT)btinternet.com), Jan 16 2005

%C A001511 Relation to C(n) = Collatz function iteration using only odd steps: a(n) is the number of right bits set in binary representation of A004767(n) (numbers of the form 4*m+3). So for m=A004767(n) it follows that there are exactly a(n) recursive steps where m<C(m). –Lambert Klasen (lambert.klasen(AT)gmx.de), Jan 23 2005

%C A001511 The ordinal transform of a sequence b_0, b_1, b_2, ... is the sequence a_0, a_1, a_2, ... where a_n is the number of times b_n has occurred in {b_0 ... b_n}.

%C A001511 Between every two instances of any positive integer m there are exactly m distinct values (1 through m–1 and one value greater than m). –Franklin T. Adams–Watters (FrankTAW(AT)Netscape.net), Sep 18 2006

%D A001511 J.–P. Allouche and J. Shallit, The ring of k–regular sequences, Theoretical Computer Sci., 98 (1992), 163–197.

%D A001511 E. R. Berlekamp, J. H. Conway and R. K. Guy, Winning Ways, Academic Press, NY, 2 vols., 2nd ed., 2001–2003; see Dim–and Dim+ on p. 98; Dividing Rulers, on pp. 436–437; The Ruler Game, pp. 469–470; Ruler Fours, Fives, ... Fifteens on p. 470.

%D A001511 Flajolet, P., Raoult, J.–C., and Vuillemin, J.; The number of registers

required for evaluating arithmetic expressions. Theoret. Comput. Sci. 9 (1979), no. 1, 99–125.

%D A001511 F. Q. Gouvea, p–Adic Numbers, Springer–Verlag, 1993; see p. 23.

%D A001511 A. M. Hinz, The Tower of Hanoi, in Algebras and combinatorics (Hong Kong, 1997), 277–289, Springer, Singapore, 1999.

%D A001511 Problem 636, Math. Mag., 40 (1967), 164–165.

%D A001511 Andrew Schloss, "Towers of Hanoi" composition, in The Digital Domain. Elektra/Asylum Records 9 60303–2, 1983. Works by Jaffe (Finale to "Silicon Valley Breakdown"), McNabb ("Love in the Asylum"), Schloss ("Towers of Hanoi"), Mattox ("Shaman"), Rush, Moorer ("Lions are Growing") and others.

%D A001511 Dinesh Thakur, J. Number Theory, 1991.